# Darmois-Skitovic theorem and its proof

Massoud BABAIE-ZADEH

January 18, 2002

**Lemma 1** *Suppose that for the functions $f_1, f_2, \ldots, f_N$, which are differentiable at any order, we have:*

$$f_1(a_1x + b_1y) + f_2(a_2x + b_2y) \cdots + f_N(a_Nx + b_Ny) = A(x) + B(y) \quad \forall x, y \quad (1)$$

*where $a_1, \ldots, a_N, b_1, \ldots, b_N$ are non-zero constants such that:*

$$a_ib_j - a_jb_i \neq 0 \qquad \forall\, i \neq j \quad (2)$$

*Then, all the functions $f_i$ are polynomials with the degree at most $N$.*

*Proof*: It is easy to seen that $A(x)$ and $B(y)$ will be differentiable at any order too. Now, suppose that there is small variations in $x$ and $y$ such that $a_Nx + b_Ny$ remains constant, that is, let:

$$\begin{aligned} x &\leftarrow x + \delta_1^{(1)} \\ y &\leftarrow y + \delta_2^{(1)} \\ a_N\delta_1^{(1)} + b_N\delta_2^{(1)} &= 0 \end{aligned} \quad (3)$$

(graphically, we are approaching the point $(x, y)$ on the line $a_Nx + b_Ny = 0$). But the arguments of all the other $f_i$'s ($i = 1, \ldots, N-1$) has changed by a small value $\epsilon_i^{(1)}$ which is not zero (because of assumption (2)). Hence by subtracting the new equation from (1) we will have:

$$\begin{aligned} \Delta_{\epsilon_1^{(1)}} f_1(a_1x + b_1y) + \Delta_{\epsilon_2^{(1)}} f_2(a_2x + b_2y) \cdots + \Delta_{\epsilon_{N-1}^{(1)}} f_{N-1}(a_{N-1}x + b_{N-1}y) \\ = A_1(x) + B_1(y) \quad \forall x, y \end{aligned} \quad (4)$$

where $\Delta_h f(x)$ is the first order difference (something like derivative) of the function $f$ at the point $x$, defined by:

$$\Delta_h f(x) = f(x + h) - f(x) \quad (5)$$

Now, we note that (4) is something like (1) but $f_N$ is disappeared. By repeating this procedure, we obtain:

$$\Delta_{\epsilon_1^{(N-1)}} \ldots \Delta_{\epsilon_1^{(2)}} \Delta_{\epsilon_1^{(1)}} f_1(a_1x + b_1y) = A_{N-1}(x) + B_{N-1}(y) \qquad \forall x, y \quad (6)$$

Repeating the procedure two more times, one for a small variation only in $x$ and one for a small variation only in $y$, we will have:

$$\Delta_{\epsilon_1^{(N+1)}} \ldots \Delta_{\epsilon_1^{(2)}} \Delta_{\epsilon_1^{(1)}} f_1(a_1x + b_1y) = 0 \qquad \forall x, y \quad (7)$$

In other words, the '$N + 1$'-th order difference of the function $f_1$ (and hence its '$N + 1$'-th order derivative) is zero, therefore it is a polynomial, and its degree is at most $N$. The proof is similar for all the other $f_i$'s. □

**Theorem 1 (Lévy-Cramer)** *Let $X_1$ and $X_2$ be two independent random variables and $Y = X_1 + X_2$. Then, if $Y$ has a Gaussian distribution, then $X_1$ and $X_2$ will be Gaussian, too.*

Recall: The characteristic function of the random variable $X$ is defined as:

$$\Phi_X(\omega) = E\left\{e^{j\omega X}\right\} \tag{8}$$

and its second characteristic function is:

$$\Psi_X(\omega) = \ln \Phi_X(\omega) \tag{9}$$

**Theorem 2 (Marcinkiewics-Dugué)** *The only random variables which have the characteristic functions of the form $e^{p(\omega)}$ where $p(\omega)$ is a polynomial, are the constant random values and Gaussian random variables (and hence the degree of p is less than or equal to 2).*

**Theorem 3 (Darmois-Skitovic)** *Let $X_1, \ldots, X_N$ be $N$ independent random variables. Let:*

$$\begin{cases} Y_1 &= a_1 X_1 + \cdots + a_N X_N \\ Y_2 &= b_1 X_1 + \cdots + b_N X_N \end{cases} \tag{10}$$

*and suppose that $Y_1$ and $Y_2$ are independent. Now, if for an $i$ we have $a_i b_i \neq 0$, then $X_i$ must be Gaussian.*

This theorem, which is the base of blind source separation (from it, the separability of linear instantaneous mixtures is obvious), states that a random variable which is not Gaussian cannot appears as a summation term in two independent random variables.

*Proof:* Without losing the generality, we can assume $a_i b_j - a_j b_i \neq 0$ for all $i \neq j$ (otherwise, we can combine two random variables to define another one, Guassianity of this random variable, proves the Gaussianity of both, because of Lévy-Cramer theorem). Now, we write:

$$\begin{aligned} \Phi_{Y_1 Y_2}(\omega_1, \omega_2) &= E\left\{e^{j(\omega_1 Y_1 + \omega_2 Y_2)}\right\} \\ &= E\left\{e^{j\sum_i (a_i\omega_1 + b_i\omega_2) X_i}\right\} \\ &= \Phi_{X_1}(a_1\omega_1 + b_1\omega_2)\Phi_{X_2}(a_2\omega_1 + b_2\omega_2)\cdots\Phi_{X_N}(a_N\omega_1 + b_N\omega_2) \end{aligned} \tag{11}$$

The last equation arises from the independence of $X_i$'s. But, independence of $Y_1$ and $Y_2$ implies that:

$$\Phi_{Y_1 Y_2}(\omega_1, \omega_2) = \Phi_{Y_1}(\omega_1)\Phi_{Y_2}(\omega_2) \tag{12}$$

and hence:

$$\Phi_{X_1}(a_1\omega_1 + b_1\omega_2)\Phi_{X_2}(a_2\omega_1 + b_2\omega_2)\cdots\Phi_{X_N}(a_N\omega_1 + b_N\omega_2) = \Phi_{Y_1}(\omega_1)\Phi_{Y_2}(\omega_2) \tag{13}$$

taking the logarithm of the both sides gives us:

$$\Psi_{X_1}(a_1\omega_1 + b_1\omega_2) + \Psi_{X_2}(a_2\omega_1 + b_2\omega_2) + \cdots + \Psi_{X_N}(a_N\omega_1 + b_N\omega_2) = \Psi_{Y_1}(\omega_1) + \Psi_{Y_2}(\omega_2) \tag{14}$$

Now if we first move all the term of the left side for them $a_i b_i = 0$ to the right side, and then apply the Lemma 1, we conclude that if for an $i$, $a_i b_i \neq 0$, then $\Psi_{X_i}$ must be a polynomial. Hence, from Marcinkiewics-Dugué theorem, it must be a Gaussian random variable.