2

CUMULANT SPECTRA OF STOCHASTIC SIGNALS

2.1 INTRODUCTION

In this chapter we are concerned primarily with the definitions and properties of moments, and cumulant spectra of stationary random processes. Although the value of stochastic signals at each instant of time is not known exactly, their higher-order statistics (moments and cumulants), when they exist, are multidimensional deterministic functions that possess special symmetry properties. We start by defining moments and a set of random variables, and by establishing their relationships. This is followed by the definition and properties of moments, and cumulant spectra of stationary random processes. Cumulant spectra of linear, non-Gaussian processes are then discussed, as well as their similarities and differences with cumulant spectra of nonlinear processes. Our primary goal in this chapter is to introduce all the important definitions and properties associated with polyspectra that can be found useful in applications of stochastic signal processing methods.

2.2 MOMENTS AND CUMULANTS

2.2.1 Definitions

Given a set of n real random variables $\{x_1, x_2, \ldots, x_n\}$, their joint moments of order $r = k_1 + k_2 + \ldots + k_n$ are given by [Papoulis, 1984]

$$\begin{aligned} \text{Mom}\ [x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_n}] &\triangleq E\{x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}\} = \\ &= (-j)^r \left. \frac{\partial^r \Phi(\omega_1, \omega_2, \ldots, \omega_n)}{\partial \omega_1^{k_1} \partial \omega_2^{k_2} \ldots \partial \omega_n^{k_n}} \right|_{\omega_1 = \omega_2 = \cdots = \omega_n = 0} \end{aligned} \tag{2.1}$$

where

$$\Phi(\omega_1,\omega_2,\ldots,\omega_n) \triangleq E\{\exp(j(\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n))\}$$

is their joint characteristic function. $E\{\cdot\}$ denotes the expectation operation. For example, for two random variables $\{x_1,x_2\}$, we have the second-order moments $\text{Mom}[x_1,x_2] = E\{x_1 \cdot x_2\}$, $\text{Mom}[x_1^2] = E\{x_1^2\}$ and $\text{Mom}[x_2^2] = E\{x_2^2\}$.

Another form of the joint characteristic function is defined as the natural logarithm [Papoulis, 1984] of $\Phi(\omega_1,\omega_2,\ldots,\omega_n)$; i.e.,

$$\tilde{\Psi}(\omega_1,\omega_2,\ldots,\omega_n) \triangleq \ln[\Phi(\omega_1,\omega_2,\ldots,\omega_n)]. \quad (2.2)$$

The joint **cumulants** (also called **semi-invariants**) of order r, $\text{Cum}[x_1^{k_1},x_2^{k_2},\ldots,x_n^{k_n}]$, of the same set of random variables, are defined as the coefficients in the Taylor expansion of the second characteristic function about zero [Shiryaev, 1960; 1963; Brillinger, 1965; Rosenblatt, 1983; 1985]; i.e.,

$$\text{Cum}[x_1^{k_1},x_2^{k_2},\ldots,x_n^{k_n}] \triangleq (-j)^r \left.\frac{\partial^r \tilde{\Psi}(\omega_1,\omega_2,\ldots,\omega_n)}{\partial\omega_1^{k_1}\partial\omega_2^{k_2}\ldots\partial\omega_n^{k_n}}\right|_{\omega_1=\omega_2=\ldots=\omega_n=0} \quad (2.3)$$

Thus, the joint can be expressed in terms of the joint moments of a set of random variables. For example, the moments

$$\begin{aligned} m_1 &= \text{Mom}[x_1] = E\{x_1\} & m_2 &= \text{Mom}[x_1,x_1] = E\{x_1^2\} \\ m_3 &= \text{Mom}[x_1,x_1,x_1] = E\{x_1^3\} & m_4 &= \text{Mom}[x_1,x_1,x_1,x_1] = E\{x_1^4\} \end{aligned}$$

of the random variable $\{x_1\}$ are related to its cumulants by

$$\begin{aligned} c_1 &= \text{Cum}[x_1] = m_1 \qquad c_2 = \text{Cum}[x_1,x_1] = m_2 - m_1^2 \\ c_3 &= \text{Cum}[x_1,x_1,x_1] = m_3 - 3m_2m_1 + 2m_1^3 \\ c_4 &= \text{Cum}[x_1,x_1,x_1,x_1] = m_4 - 4m_3m_1 - 3m_2^2 + 12m_2m_1^2 - 6m_1^4. \end{aligned} \quad (2.4)$$

These relationships can be verified by substituting

$$\Phi(\omega_1) = 1 + j\omega_1 m_1 - \frac{\omega_1^2}{2!}m_2 \cdots + \frac{(j\omega_1)^k}{k!}m_k + \cdots$$

into (2.1), (2.2), (2.3) and working out differentiations about zero. If $E\{x_1\} = m_1 = 0$, it follows that $c_2 = m_2$, $c_3 = m_3$, and $c_4 = m_4 - 3m_2^2$.

Example 2.1

Consider the three symmetric probability density functions (*pdfs*) shown in Figure

2.1; namely, Laplace, Gaussian, and Uniform. Their moments m_n of order $n = 1,2,3,4$ can be generated from [Papoulis, 1984]

$$m_n = \int_{-\infty}^{+\infty} x^n\ f(x)\quad dx$$

where $f(x)$ is the probability density function (*pdf*). From (2.1), we compute the characteristic function

$$\Phi(\omega) = \int_{-\infty}^{+\infty} \exp(j\omega x)\ f(x)\quad dx.$$

The cumulants c_n, $n = 1,2,3,4$ of the *pdfs* follow easily from the moments in (2.4). Figure 2.1 also illustrates the moments and cumulants of the *pdfs* from order first to fourth. Let us note that for the symmetric *pdfs* all m_n and c_n for n odd are identical to zero and that for the Gaussian case all cumulants c_n of order greater than second ($n > 2$) are also zero. □

Example 2.2

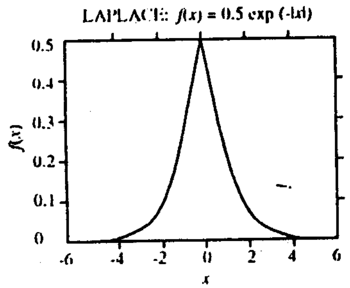
Figure 2.2 illustrates three nonsymmetric *pdfs*; i.e., Exponential, Rayleigh, and K-distribution [Watts, 1985], as well as their moments and for orders $n = 1,2,3,4$. □

2.2.2 Relationship Between Moments and Cumulants

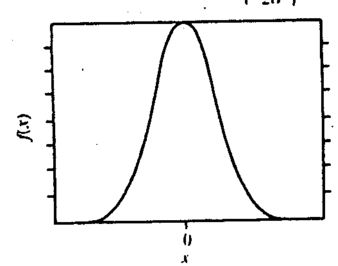
The general relationship between moments of $\{x_1,x_2,\ldots,x_n\}$ and joint cumulants $\text{Cum}[x_1,x_2,\ldots,x_n]$ of order $r = n$ is given by [Leonov and Shiryaev, 1959; Brillinger, 1965; Rosenblatt, 1985]

$$\text{Cum}[x_1,x_2,\ldots,x_n] = \sum(-1)^{p-1}(p-1)! \cdot E\{\prod_{i\in s_1} x_i\}\cdot E\{\prod_{i\in s_2} x_i\}\cdots E\{\prod_{i\in s_p} x_i\} \quad (2.5)$$

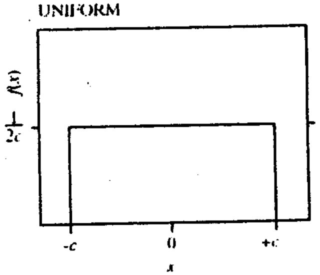
where the summation extends over all partitions $(s_1,s_2,\ldots,s_p)$, $p = 1,2,\ldots,n$, of the set of integers $(1,2,\ldots,n)$. For example, the set of integers $(1,2,3)$ can be partitioned into



n	m_n	c_n
1	0	0
2	2	2
3	0	0
4	24	12

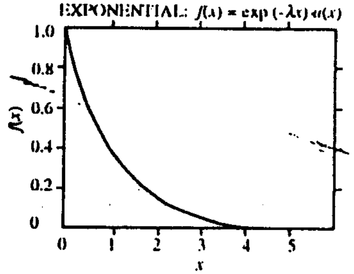


n	m_n	c_n
1	0	0
2	σ^2	σ^2
3	0	0
4	$3\sigma^4$	0

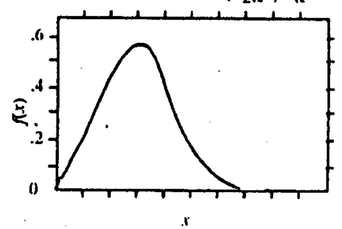


n	m_n	c_n
1	0	0
2	$\frac{c^2}{3}$	$\frac{c^2}{3}$
3	0	0
4	$\frac{c^4}{5}$	$-\frac{2c^4}{15}$

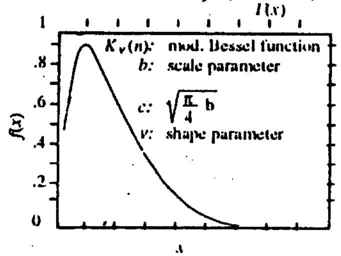
Figure 2.1 The nth-order moments and cumulants for $n = 1, 2, 3, 4$ of the Laplace, Gaussian, and Uniform Probability Density Functions (pdfs)



n	m_n	c_n
1	$\frac{1}{\lambda}$	$\frac{1}{\lambda}$
2	$\frac{2}{\lambda^2}$	$\frac{1}{\lambda^2}$
3	$\frac{6}{\lambda^3}$	$\frac{2}{\lambda^3}$
4	$\frac{24}{\lambda^4}$	$\frac{6}{\lambda^4}$



n	m_n	c_n
1	$\alpha\sqrt{\frac{\pi}{2}}$	$\alpha\sqrt{\frac{\pi}{2}}$
2	$2\alpha^2$	$\left(2 - \frac{\pi}{2}\right)\alpha^2$
3	$3\alpha^3\sqrt{\frac{\pi}{2}}$	$-3\alpha^3\pi\sqrt{\frac{\pi}{2}}$
4	$8\alpha^4$	$\frac{1}{2}\alpha^4\left[12\pi - 3\pi^2 - 8\right]$



$$m_n = \frac{1}{c^n} \cdot \frac{\Gamma\left(\nu + \frac{n}{2}\right)}{\Gamma(\nu)} \cdot \Gamma\left(\frac{n}{2} + 1\right)$$

$0.1 \le \nu \le \infty$

$\nu = \infty \rightarrow$ Rayleigh

$\nu = 0.1 \rightarrow$ spiky clutter

Figure 2.2 The nth-order moments and cumulants for $n = 1, 2, 3, 4$ of Exponential, Rayleigh, and K-distribution pdfs

$$
\begin{array}{llll}
p=1 & s_1 = \{1,2,3\} & & \\
p=2 & s_1 = \{1\}, & s_2 = \{2,3\} & \\
 & s_1 = \{2\}, & s_2 = \{1,3\} & \\
 & s_1 = \{3\}, & s_2 = \{1,2\} & \\
p=3 & s_1 = \{1\}, & s_2 = \{2\}, & s_3 = \{3\}
\end{array}
$$

and therefore (2.5) becomes

$$
\begin{aligned}
\text{Cum}[x_1,x_2,x_3] &= E\{x_1x_2x_3\} - E\{x_1\}\cdot E\{x_2x_3\} - E\{x_2\}\cdot E\{x_1x_3\} \\
&- E\{x_3\}\cdot E\{x_1x_2\} + 2E\{x_1\}\cdot E\{x_2\}\cdot E\{x_3\}.
\end{aligned} \tag{2.6}
$$

Clearly, (2.6) is identical to c_3 of (2.4) for $x_1 = x_2 = x_3$. On the other hand, if we are given $\{x_1, x_2, x_3, x_4\}$, then all possible partitions of the set of integers (1,2,3,4) is given in Table 2.1. As such, (2.5) takes the form shown in (2.7):

$$
\begin{aligned}
\text{Cum}[x_1,x_2,x_3,x_4] &= E\{x_1x_2x_3x_4\} - E\{x_1x_2\}\cdot E\{x_3x_4\} \\
&- E\{x_1x_3\}\cdot E\{x_2x_4\} - E\{x_1x_4\}\cdot E\{x_2x_3\} \\
&- E\{x_1\}\cdot E\{x_2x_3x_4\} - E\{x_2\}\cdot E\{x_1x_3x_4\} \\
&- E\{x_3\}\cdot E\{x_1x_2x_4\} - E\{x_4\}\cdot E\{x_1x_2x_3\} \\
&+ 2E\{x_1x_2\}\cdot E\{x_3\}\cdot E\{x_4\} \\
+2E\{x_1x_3\} &\cdot E\{x_2\}\cdot E\{x_4\} + 2E\{x_1x_4\}\cdot E\{x_2\}\cdot E\{x_3\} \\
+2E\{x_2x_4\} &\cdot E\{x_1\}\cdot E\{x_3\} + 2E\{x_3x_4\}\cdot E\{x_1\}\cdot E\{x_2\} \\
+2E\{x_2x_3\} &\cdot E\{x_1\}\cdot E\{x_4\} \\
-6E\{x_1\} &\cdot E\{x_2\}\cdot E\{x_3\}\cdot E\{x_4\}.
\end{aligned} \tag{2.7}
$$

Two important observations can be made from (2.7). First, (2.7) becomes identical to c_4 of (2.4) if we assume $x_1 = x_2 = x_3 = x_4$. Second, if the random variables have zero-mean (i.e., $E\{x_i\} = 0$, $i = 1,2,3,4$), then (2.7) turns out to be the well-known expression

$$
\begin{aligned}
\text{Cum}[x_1,x_2,x_3,x_4] &= E\{x_1x_2x_3x_4\} - E\{x_1x_2\}\cdot E\{x_3x_4\} \\
&- E\{x_1x_3\}\cdot E\{x_2x_4\} - E\{x_1x_4\}\cdot E\{x_2x_3\}
\end{aligned} \tag{2.8}
$$

The relationship (2.5) implies that **the computation of joint of order r requires knowledge of all moments up to order r.**

2.2.3 Properties of Moments and Cumulants

The properties of moments and cumulants may be summarized as follows [Shiryaev, 1960; 1963; Sinai, 1963; Brillinger, 1965; Rosenblatt, 1983; 1985; Brillinger and Rosenblatt, 1967; Rao and Gabr, 1984]:

TABLE 2.1 ALL POSSIBLE PARTITIONS OF SET {1,2,3,4} NEEDED FOR COMPUTATION OF $\text{Cum}[x_1,x_2,x_3,x_4]$.

p \ Partions	s_1	s_2	s_3	s_4
1	1,2,3,4	ϕ	ϕ	ϕ
	1,2	3,4	ϕ	ϕ
	1,3	2,4	ϕ	ϕ
	1,4	2,3	ϕ	ϕ
2	1	2,3,4	ϕ	ϕ
	2	1,3,4	ϕ	ϕ
	3	1,2,4	ϕ	ϕ
	4	1,2,3	ϕ	ϕ
	1,2	3	4	ϕ
	1,3	2	4	ϕ
	1,4	2	3	ϕ
3	2,4	1	3	ϕ
	3,4	1	2	ϕ
	2,3	1	4	ϕ
4	1	2	3	4

1. $\text{Mom}[a_1x_1, a_2x_2, \ldots, a_nx_n] = a_1 \ldots a_n \text{ Mom}[x_1,\ldots,x_n]$ and $\text{Cum}[a_1x_1, a_2x_2, \ldots, a_nx_n] = a_1 \ldots a_n \text{ Cum}[x_1,\ldots,x_n]$ where $(a_1, a_2, \ldots, a_n)$ are constants. This follows directly from (2.1) and (2.5).

2. Moments and cumulants are symmetric functions in their arguments, e.g., $\text{Mom}[x_1,x_2,x_3] = \text{Mom}[x_2,x_1,x_3] = \text{Mom}[x_3,x_2,x_1]$, and so on.

3. If the random variables $\{x_1, x_2, \ldots, x_n\}$ can be divided into any two or more groups which are statistically independent, their nth-order cumulant is identical to zero; i.e., $\text{Cum}[x_1,x_2,\ldots,x_n] = 0$ whereas, in general, $\text{Mom}[x_1,x_2,\ldots,x_n] \neq 0$. For example, if the two independent groups are $\{x_1, x_2 \ldots, x_\lambda\}$ and $\{x_{\lambda+1}, \ldots, x_n\}$, then their joint characteristic function is $\Phi(\omega_1,\omega_2,\ldots,\omega_n) = \Phi_1(\omega_1,\ldots,\omega_\lambda)\cdot\Phi_2(\omega_{\lambda+1},\ldots,\omega_n)$. On the other hand, their joint second characteristic function is $\tilde{\Psi}(\omega_1,\omega_2,\ldots,\omega_n) = \tilde{\Psi}_1(\omega_1,\ldots,\omega_\lambda) + \tilde{\Psi}_2(\omega_{\lambda+1},\ldots,\omega_n)$. The proof of this property easily follows if we substitute $\tilde{\Psi}(\omega_1,\ldots,\omega_n)$ and $\Phi(\omega_1,\ldots,\omega_n)$ into (2.3) and (2.1), respectively.

4. If the sets of random variables $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ are independent, then

$$\text{Cum}[x_1+y_1, x_2+y_2, \ldots, x_n+y_n] = \text{Cum}[x_1,\ldots,x_n] + \text{Cum}[y_1,\ldots,y_n]$$

whereas in general

$$\begin{aligned}\text{Mom}[x_1+y_1,\ldots,x_n+y_n] &= E\{(x_1+y_1)(x_2+y_2)\cdots(x_n+y_n)\}\\ &\neq \text{Mom}[x_1,\ldots,x_n]+\text{Mom}[y_1,\ldots,y_n].\end{aligned}$$

However, for the random variables $\{y_1, x_1, x_2, \ldots, x_n\}$ we have that

$$\text{Cum}[x_1+y_1, x_2,\ldots,x_n] = \text{Cum}[x_1,x_2,\ldots,x_n]+\text{Cum}[y_1,x_2,\ldots,x_n]$$

and

$$\text{Mom}[x_1+y_1, x_2,\ldots,x_n] = \text{Mom}[x_1,x_2,\ldots,x_n]+\text{Mom}[y_1,x_2,\ldots,x_n].$$

5. If the set of random variables $\{x_1,\ldots,x_n\}$ is jointly Gaussian, then all the information about their distribution is contained in the moments of order $n \leq 2$. Therefore, all moments of order greater than two ($n > 2$) have no new information to provide. This leads to the fact that all joint cumulants of order $n > 2$ are identical to zero for Gaussian random vectors. Hence, the cumulants of order greater than two, in some sense, measure the non-Gaussian nature (or nonnormality) of a time series.

Example 2.3

Consider the random variables

$$z_i = y_i + x_i, \qquad i = 1,2,3$$

where the joint probability density function of $\{y_1, y_2, y_3\}$ is non-Gaussian and $\{x_1, x_2, x_3\}$ is jointly Gaussian and independent from $\{y_1, y_2, y_3\}$. Let us also assume that $E\{y_i\} \neq 0$, $E\{x_i\} \neq 0$ for $i = 1, 2, 3$. From properties (4), (5) of cumulants, it follows that

$$\text{Cum}[z_1, z_2, z_3] = \text{Cum}[y_1, y_2, y_3]$$

because $\text{Cum}[x_1, x_2, x_3] = 0$. On the other hand,

$$\begin{aligned}\text{Mom}[z_1,z_2,z_3] &= E\{z_1\cdot z_2\cdot z_3\} = E\{(y_1+x_1)(y_2+x_2)(y_3+x_3)\}\\ &= \text{Mom}[y_1,y_2,y_3]+\text{Mom}[x_1,x_2,x_3]\\ &+ \text{Mom}[y_1,x_2,y_3]+\text{Mom}[x_1,y_2,y_3]+\text{Mom}[x_1,x_2,y_3]\\ &+ \text{Mom}[y_1,y_2,x_3]+\text{Mom}[y_1,x_2,x_3]+\text{Mom}[x_1,y_2,x_3].\end{aligned}$$

We see that if $E\{y_i\} = E\{x_i\} = 0$ for all i, then

$$\text{Mom}[z_1,z_2,z_3] = \text{Mom}[y_1,y_2,y_3]+\text{Mom}[x_1,x_2,x_3].$$

This simple example demonstrates one of the key motivations behind the use of cumulants in signal processing problems; namely, their ability to suppress noise (x_i) when it is additive Gaussian. □

2.2.4 Moments and Cumulants of Stationary Processes

If $\{X(k)\}$, $k = 0, \pm1, \pm2, \pm3, \ldots$ is a real stationary random process and its moments up to order n exist, then

$$\text{Mom}[X(k), X(k+\tau_1),\ldots,X(k+\tau_{n-1})] = E\{X(k)\cdot X(k+\tau_1)\cdots X(k+\tau_{n-1})\}$$

will depend only on the time differences $\tau_1, \tau_2, \ldots, \tau_{n-1}$, $\tau_i = 0, \pm1, \pm2, \ldots$ for all i. We now write the moments of a stationary random process as:

$$m_n^x(\tau_1,\tau_2,\ldots,\tau_{n-1}) \triangleq E\{X(k)\cdot X(k+\tau_1)\cdots X(k+\tau_{n-1})\}. \tag{2.9}$$

Similarly, the nth-order cumulants of $\{X(k)\}$ are $(n-1)$-dimensional functions which we now write in the form:

$$c_n^x(\tau_1,\tau_2,\ldots,\tau_{n-1}) \triangleq \text{Cum}[X(k), X(k+\tau_1),\ldots,X(k+\tau_{n-1})]. \tag{2.10}$$

Combining (2.5), (2.9), and (2.10), we obtain the following relationships between moment and cumulant sequences of $X(k)$:

1st-order cumulants:

$$c_1^x = m_1^x = E\{X(k)\} \quad \text{(mean value)} \tag{2.11}$$

2nd-order cumulants:

$$\begin{aligned}c_2^x(\tau_1) &= m_2^x(\tau_1) - (m_1^x)^2 \quad \text{(covariance sequence)}\\ &= m_2^x(-\tau_1) - (m_1^x)^2 = c_2^x(-\tau_1)\end{aligned} \tag{2.12}$$

where $m_2^x(\tau_1)$ is the autocorrelation sequence.

3rd-order cumulants:

$$c_3^x(\tau_1,\tau_2) = m_3^x(\tau_1,\tau_2) - m_1^x[m_2^x(\tau_1)+m_2^x(\tau_2)+m_2^x(\tau_2-\tau_1)] + 2(m_1^x)^3 \tag{2.13}$$

where $m_3^x(\tau_1, \tau_2)$ is the 3rd-order moment sequence. This follows if we combine (2.6) and (2.10).

4th-order cumulants: Combining (2.7) and (2.10), we get

$$\begin{aligned} c_4^x(\tau_1,\tau_2,\tau_3) &= m_4^x(\tau_1,\tau_2,\tau_3) - m_2^x(\tau_1)\cdot m_2^x(\tau_3-\tau_2) - m_2^x(\tau_2)\cdot m_2^x(\tau_3-\tau_1) \\ &- m_2^x(\tau_3)\cdot m_2^x(\tau_2-\tau_1) - m_1^x[m_3^x(\tau_2-\tau_1,\tau_3-\tau_1) \\ &+ m_3^x(\tau_2,\tau_3) + m_3^x(\tau_1,\tau_3) + m_3^x(\tau_1,\tau_2)] \\ &+ 2(m_1^x)^2[m_2^x(\tau_1) + m_2^x(\tau_2) + m_2^x(\tau_3) + m_2^x(\tau_3-\tau_1) + m_2^x(\tau_3-\tau_2) \\ &+ m_2^x(\tau_2-\tau_1)] - 6(m_1^x)^4. \end{aligned} \tag{2.14}$$

If the process $\{X(k)\}$ is zero-mean ($m_1^x = 0$), it follows from (2.12) and (2.13) that the second- and third-order cumulants are identical to the second- and third-order moments, respectively. However, to generate the fourth-order, we need knowledge of the fourth-order and second-order moments in (2.14).

The nth-order cumulant function of a non-Gaussian stationary random process $X(k)$ can be written as (for $n = 3,4$ only):

$$c_n^x(\tau_1,\tau_2,\ldots,\tau_{n-1}) = m_n^x(\tau_1,\tau_2,\ldots,\tau_{n-1}) - m_n^G(\tau_1,\tau_2,\ldots,\tau_{n-1})$$

where $m_n^x(\tau_1,\ldots,\tau_{n-1})$ is the nth-order moment function of $X(k)$ and $m_n^G(\tau_1,\tau_2,\ldots,\tau_{n-1})$ is the nth-order moment function of an equivalent Gaussian process that has the same mean value and autocorrelation sequence as $X(k)$. Clearly, if $X(k)$ is Gaussian, $m_n^x(\tau_1,\ldots,\tau_{n-1}) = m_n^G(\tau_1,\ldots,\tau_{n-1})$ and thus $c_n^x(\tau_1,\ldots,\tau_{n-1}) = 0$. Note, however, that this is only true of orders $n = 3$ and 4.

2.2.5 Variance, Skewness, and Kurtosis Measures

By putting $\tau_1 = \tau_2 = \tau_3 = 0$ in (2.12), (2.13), (2.14) and assuming $m_1^x = 0$ we get

$$\begin{aligned} \gamma_2^x &= E\{X(k)^2\} = c_2^x(0) && \text{(variance)} \\ \gamma_3^x &= E\{X^3(k)\} = c_3^x(0,0) && \text{(skewness)} \\ \gamma_4^x &= E\{X^4(k)\} - 3[\gamma_2^x]^2 = c_4^x(0,0,0) && \text{(kurtosis).} \end{aligned} \tag{2.15}$$

Normalized kurtosis is defined as $\gamma_4^x/[\gamma_2^x]^2$. Equation (2.15) gives the variance, skewness, and kurtosis measures in terms of cumulant lags.

2.2.6 Time-Reversible Process

A process $\{X(k)\}$ is said to be time-reversible if the probability structure of $\{X(-k)\}$ is the same as that of $\{X(k)\}$. This implies that

$$c_n^x(\tau_1,\tau_2,\ldots,\tau_{n-1}) = c_n^x(-\tau_1,-\tau_2,\ldots,-\tau_{n-1}) \tag{2.16}$$

for all integer values of $\tau_1,\tau_2,\ldots,\tau_{n-1}$. If the nth-order cumulants of a process satisfy (2.16), the process is time-reversible [Rao and Gabr, 1984]. Clearly, Gaussian processes ($n = 2$) are time-reversible due to the symmetry property of the covariance function; i.e., $c_2^x(\tau_1) = c_2^x(-\tau_1)$. Remember that higher-order ($n > 2$) cumulants of a Gaussian process are all zero.

2.2.7 Cross-Cumulants

Suppose we are given stationary real random processes $\{X(k)\}$, $\{Y(k)\}$, $\{Z(k)\}$, $k = 0, \pm 1, \pm 2, \ldots$. Their cross-cumulants may be defined as follows.

2nd-order cross-cumulants

$$\begin{aligned} c_{xy}(\tau_1) &= \text{Cum}[X(k), Y(k+\tau_1)] \\ c_{yz}(\tau_1) &= \text{Cum}[Y(k), Z(k+\tau_1)]. \end{aligned} \quad \text{(cross} - \text{covariances)} \tag{2.17}$$

Thus, if the processes are zero-mean, then $c_{xy}(\tau_1) = E\{X(k)Y(k+\tau_1)\}$ and $c_{yz}(\tau_1) = E\{Y(k)Z(k+\tau_1)\}$.

3rd-order cross-cumulants

$$\begin{aligned} c_{xyz}(\tau_1,\tau_2) &= \text{Cum}[X(k), Y(k+\tau_1), Z(k+\tau_2)] \\ &= E\{(X(k)-m_x)(Y(k+\tau_1)-m_y)(Z(k+\tau_2)-m_z)\} \end{aligned} \tag{2.18}$$

where $m_x = E\{X(k)\}$, $m_y = E\{Y(k)\}$, and $m_z = E\{Z(k)\}$. For zero-mean processes we have

$$c_{xyz}(\tau_1,\tau_2) = E\{X(k)Y(k+\tau_1)Z(k+\tau_2)\}.$$

Additional cross-cumulants are, for example

$$\begin{aligned} c_{xyy}(\tau_1, \tau_2) &= \text{Cum}[X(k), Y(k+\tau_1), Y(k+\tau_2)] \\ c_{xyx}(\tau_1, \tau_2) &= \text{Cum}[X(k), Y(k+\tau_1), X(k+\tau_2)] \end{aligned} \tag{2.19}$$

So, the nth-order cross-cumulant sequence of stationary processes $\{X_i(k)\}$, $i = 1, 2, \ldots, n$ is defined as

$$c_{x_1x_2\ldots x_n}(\tau_1, \tau_2, \ldots, \tau_{n-1}) \triangleq \text{Cum}[X_1(k), X_2(k+\tau_1), \ldots, X_n(k+\tau_{n-1})]. \tag{2.20}$$

These quantities become useful in nonlinear system identification problems when we have access to input and output measurements [Brillinger, 1965; Brillinger and Rosenblatt, 1967; Brillinger, 1977]. Essentially, we will use $c_{x_1,x_2,\ldots,x_n}(\tau_1, \ldots, \tau_{n-1})$ to test for the nonlinearity of a function of order $n-1$.

Example 2.4

Consider the narrow-band process

$$Z(k) = X(k) \cos(\omega_o k) + Y(k) \sin(\omega_o k)$$

where $X(k), Y(k)$ are independent stationary random processes with $E\{X(k)\} = E\{Y(k)\} = 0$, $m_2^x(\tau) = E\{X(k)X(k+\tau)\} = m_2^y(\tau)$, and $m_3^x(\tau_1, \tau_2) = E\{X(k)X(k+\tau_1)X(k+\tau_2)\} = m_3^y(\tau_1, \tau_2)$.

We now evaluate the autocorrelation and third-order moment sequence of $Z(k)$. The second-order moment is:

$$\begin{aligned} \text{Mom}[Z(k), Z(k+\tau)] &= E\{(X(k)\cos(\omega_o k) + Y(k)\sin(\omega_o k)) \cdot \\ &\quad (X(k+\tau)\cos(\omega_o(k+\tau)) + Y(k+\tau)\sin(\omega_o(k+\tau)))\} = \\ &= m_2^x(\tau) \cos(\omega_o \tau) = m_2^z(\tau) \end{aligned}$$

which is independent of k. Thus, $\{Z(k)\}$ is a wide-sense stationary random process. However, the third-order moments are

$$\begin{aligned} &\text{Mom}[Z(k), Z(k+\tau_1), Z(k+\tau_2)] \\ &= m_3^x(\tau_1, \tau_2)[\cos(\omega_o k)\cos(\omega_o(k+\tau_1))\cos(\omega_o(k+\tau_2)) \\ &+ \sin(\omega_o k)\sin(\omega_o(k+\tau_1))\sin(\omega_o(k+\tau_2))] \end{aligned}$$

and the quantities in square brackets are dependent on k for τ_1, τ_2. Hence $\{Z(k)\}$ is nonstationary in its third-order statistics. □

2.2.8 Ergodicity and Moments

According to Papoulis [1984], a process $\{X(k)\}$ is ergodic in the most general form if, with probability one, all its moments can be determined from a single observation. In other words, the expected values $E\{\cdot\}$ (or ensemble averages) can be replaced by time averages; i.e.,

$$\begin{aligned} &E\{X(k) \cdot X(k+\tau_1) \cdots X(k+\tau_{n-1})\} = < X(k) \cdots X(k+\tau_{n-1}) > = \\ &\lim_{M \to \infty} \frac{1}{2M+1} \sum_{k=-M}^{+M} X(k)\, X(k+\tau_1) \cdots X(k+\tau_{n-1}) \end{aligned} \tag{2.21}$$

where $< \cdot >$ is the time-average operator which has the same properties as the ensemble average operation $E\{\cdot\}$ if the process is ergodic [Sinai, 1963].

We see from (2.21) that time-averages of higher-order moments are functions of infinitely many random variables and, therefore, can be viewed as random variables themselves. What ergodicity implies is that the time averages of all possible sample sequences are equal to the same constant which, in turn, equals the ensemble average. Clearly, a process might be ergodic for certain higher-order moments and not for others [Papoulis, 1984].

We shall not discuss here the various criteria for ergodicity related to time averages of higher-order moments. Throughout this chapter we assume that if the process is ergodic, then (2.21) holds for all orders up to n. This implies that nth-order cumulants exist and can be generated from (2.5).

In practice, when we are given a finite length single realization of an ergodic process, i.e., $X(k)$, $k = -M, \ldots, 0, \ldots, +M$, we cannot compute the limits of (2.21) but the **estimates**

$$< X(k) \cdots X(k+\tau_{n-1}) >_M = \frac{1}{2M+1} \sum_{k=-M}^{+M} \cdot X(k) \cdots X(k+\tau_{n-1}). \tag{2.22}$$

The estimation of higher-order moments and thus of a stochastic process is a problem of statistics which will be examined in Chapter 4.

2.3 CUMULANT SPECTRA

Suppose that the process $\{X(k)\}$, $k = 0, \pm1, \pm2, \ldots$ is real, strictly stationary, with nth-order cumulant sequence $c_n^x(\tau_1, \ldots, \tau_{n-1})$ defined by (2.10).

2.3.1 Definition

Assuming that the cumulant sequence satisfies the condition

$$\sum_{\tau_1=-\infty}^{+\infty} \cdots \sum_{\tau_{n-1}=-\infty}^{+\infty} \left| c_n^x(\tau_1, \ldots, \tau_{n-1}) \right| < \infty,$$

or the condition

$$\sum_{\tau_1=-\infty}^{+\infty} \cdots \sum_{\tau_{n-1}=-\infty}^{+\infty} (1 + |\tau_j|) \left| c_n^x(\tau_1, \ldots, \tau_{n-1}) \right| < \infty. \quad (2.23)$$

for $j = 1, 2, \ldots, n-1$, the nth-order cumulant spectrum $C_n^x(\omega_1, \ldots, \omega_{n-1})$ of $\{X(k)\}$ exists, is continuous, and is defined as the $(n-1)$-dimensional Fourier transform of the nth-order cumulant sequence; e.g., [Brillinger, 1965; Rosenblatt, 1983; 1985]. Note that (2.23) describe the usual conditions for a Fourier transform to be well defined. The nth-order cumulant spectrum is thus defined:

$$C_n^x(\omega_1, \omega_2, \ldots, \omega_{n-1}) = \sum_{\tau_1=-\infty}^{+\infty} \cdots \sum_{\tau_{n-1}=-\infty}^{+\infty} c_n^x(\tau_1, \tau_2, \ldots, \tau_{n-1})$$

$$\exp\{-j(\omega_1\tau_1 + \omega_2\tau_2 + \cdots + \omega_{n-1}\tau_{n-1})\} \quad (2.24)$$

$|\omega_i| \leq \pi$ for $i = 1, 2, \ldots, n-1$ and $|\omega_1 + \omega_2 + \ldots + \omega_{n-1}| \leq \pi$.

In general, $C_n^x(\omega_1, \omega_2, \ldots, \omega_{n-1})$ is complex, i.e., it has magnitude and phase

$$C_n^x(\omega_1, \ldots, \omega_{n-1}) = \left|C_n^x(\omega_1, \ldots, \omega_{n-1})\right| \exp\{j\Psi_n^x(\omega_1, \ldots, \omega_{n-1})\}. \quad (2.25)$$

The cumulant spectrum is also periodic with period 2π, i.e.,

$$C_n^x(\omega_1, \ldots, \omega_{n-1}) = C_n^x(\omega_1 + 2\pi, \ldots, \omega_{n-1} + 2\pi).$$

The notion of considering a spectral representation for a cumulant function as shown in (2.24) (cumulant spectrum) is acknowledged to be due to Kolmogorov [Shiryaev,

1960; 1963]. The term "higher-order spectrum" is due to Brillinger [1965] and Akaike [1966]. The term "polyspectra" is due to Brillinger [1965].

2.3.2 Alternative Definition

The physical significance of cumulant spectra becomes apparent when expressed in terms of the components $dZ(\omega)$ of the Fourier-Stieltjes representation of $\{X(k)\}$ (Cramer spectral representation) [Rosenblatt, 1983; 1985].

$$X(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\{j\omega k\}\, dZ(\omega) \quad (2.26)$$

for all k, where

$$E\{dZ(\omega)\} = 0$$

$$\text{Cum}[dZ(\omega_1), dZ(\omega_2), \ldots, dZ(\omega_n)] = \begin{cases} C_n^x(\omega_1, \ldots, \omega_{n-1})d\omega_1 \cdots d\omega_{n-1}, \text{ for} \\ \omega_1 + \omega_2 + \ldots + \ldots \omega_{n-1} + \omega_n = 0 \\ 0, \text{ for} \\ \omega_1 + \omega_2 + \ldots + \omega_{n-1} + \omega_n \neq 0. \end{cases} \quad (2.27)$$

It is therefore apparent that the cumulant spectrum of order n represents the cumulant contribution of n Fourier components, the sum of whose frequencies equal zero. Although the cumulant spectrum of order n is a function of $n-1$ variables $\omega_1, \omega_2, \ldots, \omega_{n-1}$, it should be kept in mind that there is a hidden variable $\omega_n = -\omega_1 - \ldots - \omega_{n-1}$ in (2.27) [Rosenblatt, 1983].

2.3.3 Special Cases of Cumulant Spectra

The power spectrum, bispectrum, and trispectrum are special cases of the nth-order cumulant spectrum defined by (2.24) [Brillinger and Rosenblatt, 1967 a and b].

Power Spectrum: n=2

$$C_2^x(\omega) = \sum_{\tau=-\infty}^{+\infty} c_2^x(\tau) \exp\{-j(\omega\tau)\}, \quad (2.28)$$

$|\omega| \leq \pi$ where $c_2^x(\tau)$ is the covariance sequence of $\{X(k)\}$ given by (2.12). If the process $\{X(k)\}$ is zero-mean, then (2.28) becomes the Wiener-Khintchine identity.

From (2.12) and (2.28) we have

$$\begin{aligned} c_2^x(\tau) &= c_2^x(-\tau) \\ C_2^x(\omega) &= C_2^x(-\omega) \\ C_2^x(\omega) &\geq 0 \quad \text{(real, nonnegative function)} \end{aligned} \tag{2.29}$$

Bispectrum: n=3

$$C_3^x(\omega_1,\omega_2) = \sum_{\tau_1=-\infty}^{+\infty}\sum_{\tau_2=-\infty}^{+\infty} c_3^x(\tau_1,\tau_2)\exp\{-j(\omega_1\tau_1+\omega_2\tau_2)\} \tag{2.30}$$

$$|\omega_1|\leq\pi, |\omega_2|\leq\pi, |\omega_1+\omega_2|\leq\pi$$

where $c_3^x(\tau_1,\tau_2)$ is the third-order cumulant sequence of $\{X(k)\}$ described by (2.13). Important symmetry conditions follow from the properties of moments and (2.13):

$$\begin{aligned} c_3^x(\tau_1,\tau_2) &= c_3^x(\tau_2,\tau_1) = c_3^x(-\tau_2,\tau_1-\tau_2) \\ &= c_3^x(\tau_2-\tau_1,-\tau_1) = c_3^x(\tau_1-\tau_2,-\tau_2) \\ &= c_3^x(-\tau_1,\tau_2-\tau_1). \end{aligned} \tag{2.31}$$

As a consequence, knowing the third-order cumulants in any of the six sectors, I through VI, shown in Figure 2.3(a), would enable us to find the entire third-order cumulant sequence. These sectors include their boundaries so that, for example, sector I is an infinite wedge bounded by the lines $\tau_1 = 0$, and $\tau_1 = \tau_2$; $\tau_1,\tau_2 \geq 0$.

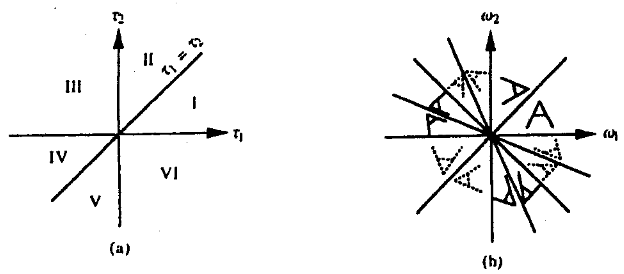


Figure 2.3 (a) Symmetry regions of third-order moments. (b) Symmetry regions of the bispectrum.

The definition of the bispectrum in (2.30) and the properties of third-order cumulants in (2.31) give

$$\begin{aligned} C_3^x(\omega_1,\omega_2) &= C_3^x(\omega_2,\omega_1) \\ &= C_3^{x*}(-\omega_2,-\omega_1) = C_3^x(-\omega_1-\omega_2,\omega_2) \\ &= C_3^x(\omega_1,-\omega_1-\omega_2) = C_3^x(-\omega_1-\omega_2,\omega_1) \\ &= C_3^x(\omega_2,-\omega_1-\omega_2). \end{aligned} \tag{2.32}$$

Thus knowledge of the bispectrum in the triangular region $\omega_2 \geq 0$, $\omega_1 \geq \omega_2$, $\omega_1+\omega_2 \leq \pi$ shown in Figure 2.3(b) is enough for a complete description of the bispectrum. For real processes, the bispectrum has 12 symmetry regions.

Trispectrum: n=4

$$C_4^x(\omega_1,\omega_2,\omega_3) = \sum_{\tau_1=-\infty}^{+\infty}\sum_{\tau_2=-\infty}^{+\infty}\sum_{\tau_3=-\infty}^{+\infty} c_4^x(\tau_1,\tau_2,\tau_3)\cdot\exp\{-j(\omega_1\tau_1+\omega_2\tau_2+\omega_3\tau_3)\} \tag{2.33}$$

$$|\omega_1|\leq\pi, |\omega_2|\leq\pi, |\omega_3|\leq\pi, |\omega_1+\omega_2+\omega_3|\leq\pi$$

where $c_4^x(\tau_1,\tau_2,\tau_3)$ is the fourth-order cumulant sequence given by (2.14).

From the definition (2.14) of fourth-order cumulants, a lot of symmetry properties can be derived for the trispectrum, similar to those given in (2.32) for the bispectrum. For example, since the moments and are symmetric functions in their arguments, we have

$$\begin{aligned} C_4^x(\omega_1,\omega_2,\omega_3) &= C_4^x(\omega_2,\omega_1,\omega_3) = C_4^x(\omega_3,\omega_2,\omega_1) \\ &= C_4^x(\omega_1,\omega_3,\omega_2) = C_4^x(\omega_2,\omega_3,\omega_1) \\ &= C_4^x(\omega_3,\omega_1,\omega_2) = \text{etc.} \end{aligned} \tag{2.34}$$

Pflug et al. [1992] point out that the trispectrum of real processes has 96 symmetry regions.

2.3.4 Variance, Skewness, and Kurtosis Measures

Inverse Fourier transformation on (2.24) yields

$$c_n^x(\tau_1, \tau_2, \ldots, \tau_{n-1}) = \frac{1}{(2\pi)^{y-1}} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} \cdots \int_{-\pi}^{+\pi} C_n^x(\omega_1, \ldots, \omega_{n-1}) \exp\{j(\omega_1\tau_1 + \ldots + \omega_{n-1}\tau_{n-1})\} d\omega_1 \cdots d\omega_{n-1}. \quad (2.35)$$

By choosing $n = 2,3,4$ and setting $(\tau_i) = 0, \quad i = 1,2,\ldots,n-1$, we get

$$\begin{aligned} c_2^x(0) &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} C_2^x(\omega) d\omega \quad \text{(variance } \gamma_2^x) \\ c_3^x(0,0) &= \frac{1}{(2\pi)^2} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} C_3^x(\omega_1, \omega_2) d\omega_1 d\omega_2 \quad \text{(skewness } \gamma_3^x) \\ c_4^x(0,0,0) &= \frac{1}{(2\pi)^3} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} C_4^x(\omega_1, \omega_2, \omega_3) d\omega_1 d\omega_2 d\omega_3 \quad \text{(kurtosis } \gamma_4^x) \end{aligned} \quad (2.36)$$

which are the measures also described in (2.15). However, the measures in (2.36) are expressed in terms of cumulant spectra.

2.3.5 Time-Reversible Processes

An alternative way to (2.16) of testing whether a given stationary random process $\{X(k)\}$ is time-reversible is by examining the imaginary part of its cumulant spectra [Rao and Gabr, 1984]. From the condition of time-reversibility (2.16) and the definition of cumulant spectra (2.24), it follows that a process $\{X(k)\}$ is said to be time-reversible if the imaginary parts of all its cumulant spectra are identically zero. Since a Gaussian process has all its cumulant spectra of order $n > 2$ equal to zero and the imaginary part of its 2nd-order spectrum (power spectrum) is also zero, we conclude that a Gaussian process is time-reversible.

2.3.6 Non-Gaussian White Noise Processes

If $\{W(k)\}$ is a stationary non-Gaussian process with $E\{W(k)\} = 0$ and with nth-order cumulant sequence

$$\begin{aligned} c_n^w(\tau_1, \tau_2, \ldots, \tau_{n-1}) &= \text{Cum}[W(k), W(k+\tau_1), \ldots, W(k+\tau_{n-1})] \\ &= \gamma_n^w \, \delta(\tau_1, \tau_2, \ldots, \tau_{n-1}) \end{aligned} \quad (2.37)$$

where γ_n^w is constant and $\delta(\tau_1, \tau_2, \ldots, \tau_{n-1})$ is the $(n-1)$-dimensional Kronecker delta function, then $\{W(k)\}$ is said to be white of order n [Brillinger, 1965; Giannakis and Mendel, 1989]. Of course, one need not assume all γ_n^w are finite. The cumulants γ_n^w cannot all be zero (assuming they exist) for $n \geq 2$. Combining (2.37) and (2.24) we obtain

$$C_n^w(\omega_1, \ldots, \omega_{n-1}) = \gamma_n^w \quad (2.38)$$

which is a flat spectrum for all frequencies. Hence, consideration of (2.36), (2.37), and (2.38) leads to the following important special cases of white noise higher-order spectra.

$$\begin{aligned} C_2^w(\omega) &= \gamma_2^w \quad \text{(Power Spectrum)} \\ C_3^w(\omega_1, \omega_2) &= \gamma_3^w \quad \text{(Bispectrum)} \\ C_4^w(\omega_1, \omega_2, \omega_3) &= \gamma_4^w \quad \text{(Trispectrum)} \end{aligned} \quad (2.39)$$

where γ_2^w is the variance, γ_3^w - the skewness, and γ_4^w - the kurtosis of $\{W(k)\}$.

Example 2.5

Consider the simple example [Lii and Rosenblatt, 1988]

$$X(k) = W(k) - W(k-1), \quad k = 0, \pm 1, \pm 2, \ldots$$

where $\{W(k)\}$ are independent, identically distributed (i.i.d.) with $E\{W(k)\} = 0$, $E\{W^2(k)\} = 1$ and $E\{W^3(k)\} = 1$. The covariance sequence of $\{X(k)\}$ is given by

$$\begin{aligned} c_2^x(\tau) = m_2^x(\tau) &= E\{X(k)X(k+\tau)\} \\ &= E\{(W(k) - W(k-1))(W(k+\tau) - W(k+\tau-1))\} \\ &= 2\delta(\tau) - \delta(\tau-1) - \delta(\tau+1) \end{aligned}$$

where $\delta(\tau)$ is the Kronecker delta function. Thus,

$$c_2^x(\tau) = \begin{cases} 2, & \tau = 0 \\ -1, & \tau = 1, \tau = -1 \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, the third-order cumulants of $\{X(k)\}$ are computed from

$$\begin{aligned} c_3^x(\tau_1,\tau_2) = m_3^x(\tau_1,\tau_2) &= E\{X(k)X(k+\tau_1)X(k+\tau_2)\} \\ &= E\{(W(k)-W(k-1))(W(k+\tau_1)-W(k+\tau_1-1)) \\ &\qquad (W(k+\tau_2)-W(k+\tau_2-1))\} \end{aligned}$$

which leads to

$$\begin{aligned} c_3^x(\tau_1,\tau_2) = &- \delta(\tau_1-1,\tau_2)+\delta(\tau_1-1,\tau_2-1)-\delta(\tau_1,\tau_2-1) \\ &+ \delta(\tau_1+1,\tau_2)+\delta(\tau_1+1,\tau_2+1)-\delta(\tau_1,\tau_2+1). \end{aligned}$$

Figure 2.4 illustrates the covariance and third-order cumulant sequences of $\{X(k)\}$. It is important to note that although the skewness $c_3^x(0,0,0)=\gamma_3^x=0$, the third-order cumulants $c_3^x(\tau_1,\tau_2)$ are generally different from zero.

The power spectrum of the random process is given by

$$\begin{aligned} C_2^x(\omega) &= \sum_{\tau=-1}^{+1} c_2^x(\tau)\exp(-j\omega\tau) \\ &= (2-2\cos\omega) \end{aligned}$$

whereas its bispectrum is given by

$$\begin{aligned} C_3^x(\omega_1,\omega_2) &= \Big(-e^{-j(\omega_1)}+e^{-j(\omega_1+\omega_2)}-e^{-j(\omega_2)}+e^{+j(\omega_1)} \\ &\quad - e^{+j(\omega_1+\omega_2)}+e^{+j(\omega_2)}\Big) \\ &= \Big(2j\sin\omega_1+2j\sin\omega_2-2j\sin(\omega_1+\omega_2)\Big) \\ &= 2j\Big(\sin\omega_1+\sin\omega_2-\sin(\omega_1+\omega_2)\Big). \end{aligned}$$

We observe that the real part of the bispectrum is zero and the imaginary part equals $2(\sin\omega_1+\sin\omega_2-\sin(\omega_1+\omega_2))$. Figure 2.4 also illustrates the power spectrum and the bispectrum of $\{X(k)\}$. This example illustrates that **zero skewness does not necessarily imply zero bispectrum** because the skewness of a signal contributes to the real part of the bispectrum only. □

2.3.7 One-Dimensional Slices of Cumulants and their Cumulant Spectra

Since higher-order cumulant spectra are multidimensional functions, their computation may be impractical in some applications due to excessive number crunching. Nagata [1970] suggested the use of certain 1-d slices of multidimensional cumulant sequences, and their 1-d Fourier transforms, as ways of extracting useful information from higher-order statistics (or moments) of non-Gaussian stationary processes.

Consider a non-Gaussian process $\{X(k)\}$ with third-order cumulants given by (2.13); i.e.,

$$c_3^x(\tau_1,\tau_2)=\mathrm{Cum}\{X(k),\ X(k+\tau_1),\ X(k+\tau_2)\}. \tag{2.40}$$

One-dimensional slices of $c_3^x(\tau_1,\tau_2)$ can be defined as shown in (2.41):

$$\begin{aligned} r_{2,1}^x(\tau) &\triangleq \mathrm{Cum}\{X(k),X(k),X(k+\tau)\}=c_3^x(0,\tau) \\ r_{1,2}^x(\tau) &\triangleq \mathrm{Cum}\{X(k),X(k+\tau),X(k+\tau)\}=c_3^x(\tau,\tau) \\ r_{1,2}(\tau) &\triangleq r_{1,2}(-\tau) \end{aligned} \tag{2.41}$$

which represent two straight lines with slopes 90° and 45°, respectively. Furthermore, we can define

$$\begin{aligned} s_{2,1}^x(\tau) &\triangleq \frac{1}{2}[r_{2,1}^x(\tau)+r_{1,2}^x(\tau)], \\ q_{2,1}^x(\tau) &\triangleq \frac{1}{2}[r_{2,1}^x(\tau)-r_{1,2}^x(\tau)] \end{aligned} \tag{2.42}$$

which correspond to even and odd functions, respectively. If we define as 1-d spectrum

$$R_{2,1}^x(\omega)\triangleq\sum_{\tau=-\infty}^{+\infty} r_{2,1}^x(\tau)\exp(-j\omega\tau) \tag{2.43}$$

it follows from (2.42) and (2.43) that

$$R_{2,1}^x(\omega)=\sum_{\tau=-\infty}^{+\infty}\{s_{2,1}^x(\tau)\cos(\omega\tau)-jq_{2,1}^x(\tau)\sin(\omega\tau)\}. \tag{2.44}$$

Since $s_{2,1}^x(0)=\gamma_3^x$ and $q_{2,1}^x(0)=0$, the *effective contribution to the skewness comes only from the real and symmetrical part.* Furthermore, from (2.41), (2.43), and (2.30), we obtain the relation between $R_{2,1}^x(\omega)$ and the bispectrum $C_3^x(\omega_1,\omega_2)$; viz.

$$R_{2,1}^x(\omega)=\frac{1}{2\pi}\int_{-\pi}^{+\pi} C_3^x(\omega,\sigma)d\sigma, \tag{2.45}$$

which represents the integrated bispectrum along a frequency line. Nagata [1970] points out that because the real part of $C_3^x(\omega_1,\omega_2)$ is not positive definite, even if $R_{2,1}^x(\omega_o)$ is very small, we cannot conclude that the value of $R_{2,1}^x(\omega)$ at $\omega=\omega_o$

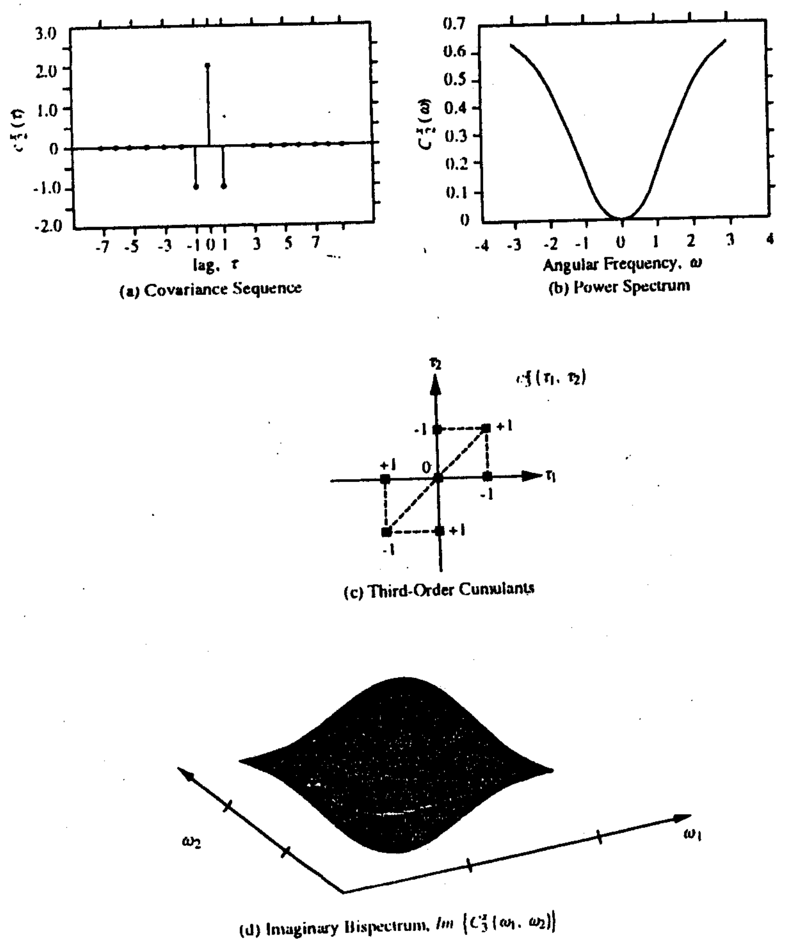


Figure 2.4 (a) The covariance sequence, (b) power spectrum, (c) third-order cumulants, and (d) bispectrum of the random process $X(k) = W(k) - W(k-1)$. Note that the real bispectrum is zero.

does not play a major role in determining the skewness. Gurbatov et al. [1987] point out that the imaginary part of $R_{2,1}^x(\omega)$ corresponds to "energy transfer" in the spectrum. A positive value of $I_m[R_{2,1}^x(\omega_o)]$ indicates energy transfer to the frequency $\omega = \omega_o$, and a negative value indicates the leakage of energy from ω_o.

If we now consider the fourth-order cumulants of $\{X(k)\}$ given by (2.14), we can define the following 1-d slices:

$$\begin{aligned} r_{3,1}(\tau) &\triangleq \mathrm{Cum}[X(k), X(k), X(k), X(k+\tau)] = c_4^x(0,0,\tau) = c_4^x(0,\tau,0) = c_4^x(\tau,0,0) \\ r_{2,2}(\tau) &\triangleq \mathrm{Cum}[X(k), X(k), X(k+\tau), X(k+\tau)] = c_4^x(0,\tau,\tau) = c_4^x(\tau,0,\tau) \\ &= c_4^x(\tau,\tau,0) \\ r_{1,3}(\tau) &\triangleq \mathrm{Cum}[X(k), X(k+\tau), X(k+\tau), X(k+\tau)] = c_4^x(\tau,\tau,\tau) \end{aligned} \tag{2.46}$$

and their functions

$$\begin{aligned} s_{3,1}^x(\tau) &= \frac{1}{2}[r_{3,1}(\tau) + r_{1,3}(\tau)] \quad \text{(even)} \\ q_{3,1}^x(\tau) &= \frac{1}{2}[r_{3,1}(\tau) - r_{1,3}(\tau)] \quad \text{(odd)} \\ s_{2,2}^x(\tau) &= r_{2,2}(\tau). \quad \text{(even)} \end{aligned} \tag{2.47}$$

Useful 1-d spectra based on 4-th order cumulants (2.46) are the following:

$$\begin{aligned} R_{3,1}^x(\omega) &= \sum_{\tau=-\infty}^{+\infty} r_{3,1}^x(\tau)\exp(-j\omega\tau) \\ &= \sum_{\tau=-\infty}^{+\infty} \{s_{3,1}^x(\tau)\cos(\omega\tau) + jq_{3,1}^x(\tau)\sin(\omega\tau)\} \end{aligned} \tag{2.48}$$

and

$$\begin{aligned} R_{2,2}^x(\omega) &= \sum_{\tau=-\infty}^{+\infty} r_{2,2}^x(\tau)\exp(-j\omega\tau) \\ &= \sum_{\tau=-\infty}^{+\infty} s_{2,2}^x(\tau)\cos(\omega\tau). \end{aligned} \tag{2.49}$$

Since $s_{2,2}^x(\tau)$ is an even sequence, its resulting spectrum $R_{2,2}^x(\omega)$ is real.

Combining (2.48), (2.46), and (2.33), we obtain the relation between $R_{3,1}^x(\omega)$ and $C_4^x(\omega_1, \omega_2, \omega_3)$. That is

$$R_{3,1}^x(\omega) = \frac{1}{(2\pi)^2} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} C_4^x(\omega, \sigma_1, \sigma_2)\, d\sigma_1\, d\sigma_2 \tag{2.50}$$

where $C_4^x(\omega, \sigma_1, \sigma_2)$ can be replaced by $C_4^x(\sigma_1, \omega, \sigma_2)$ or $C_4^x(\sigma_1, \sigma_2, \omega)$. Because $q_{3,1}^x(0) = 0$, the net contribution to kurtosis comes from $s_{3,1}^x(\tau)$ and thus from the real part of $R^x_{3,1}(\omega)$. From (2.49), (2.46), and (2.33) we obtain

$$R^x_{2,2}(\omega_1+\omega_2) = \frac{1}{2}\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi} C^x_4(\omega_1,\omega_2,\omega_3)\, d(\omega_1-\omega_2)\, d\omega_3. \qquad (2.51)$$

Therefore, $R^x_{2,2}(\omega_o)$ represents the integrated trispectrum on the plane $\omega_1+\omega_2=\omega_o$ or equivalently on the planes $\omega_1+\omega_3=\omega_o$ or $\omega_2+\omega_3=\omega_o$.

Example 2.6

Let us consider the process $\{X(k)\}$ described in Example 2.5. One-dimensional slices of its third-order cumulant sequence (shown in Figure 2.4) are obtained by utilizing (2.41); i.e.,

$$\begin{aligned} r_{2,1}(\tau) &= -\delta(\tau)+\delta(\tau+1) \\ r_{1,2}(\tau) &= \delta(\tau)-\delta(\tau+1). \end{aligned}$$

From these expressions and (2.42) we obtain

$$\begin{aligned} s^x_{2,1}(\tau) &= 0 \\ q^x_{2,1}(\tau) &= -\delta(\tau-1)+\delta(\tau+1) \end{aligned}$$

which is consistent with the fact that the real part of the bispectrum of $\{X(k)\}$ is zero. Applying to this example equation (2.43) or (2.44) we obtain

$$\begin{aligned} R^x_{2,1}(\omega) &= -j(-\sin(\omega))+\sin(-\omega) \\ &= 2j\sin\omega. \end{aligned}$$

This result is verified if we substitute in (2.45) the bispectrum of $\{X(k)\}$; i.e.,

$$C^x_3(\omega,\sigma) = 2j(\sin\omega+\sin\sigma-\sin(\omega+\sigma))$$

and perform the integration

$$\begin{aligned} R^x_{2,1}(\omega) &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} C^x_3(\omega,\sigma)\, d\sigma \\ &= 2j\sin\omega. \end{aligned}$$

Figure 2.5 illustrates the 1-d cumulant slices and $R^x_{2,1}(\omega)$ of the process $\{X(k)\}$.□

2.3.8 Why Cumulant Spectra and not Moment Spectra?

Cumulant spectra can be found more useful in the processing of random signals than moment spectra. The reason is threefold: (a) cumulant spectra of order $n>2$

are zero if the process is Gaussian and nonzero cumulant spectra provide a measure of extent of non-Gaussianity; (b) cumulants provide a suitable measure of extent of statistical dependence in time series; (c) the cumulant spectrum of the sum of two independent, nonzero mean, stationary random processes equals the sum of their individual cumulant spectra. However, this latter property does not hold in the case of moment spectra. Finally, Brillinger [1965] points out that ergodicity assumptions are met more easily in estimating cumulants rather than moments.

2.3.9 The nth-Order Coherency Function

A normalized cumulant spectrum or the nth-order coherency index is a function that combines two completely different entities, namely, the cumulant spectrum of order n, $C^x_n(\omega_1,\ldots,\omega_{n-1})$ and the power spectrum $C^x_2(\omega)$ of a process. The nth-order coherency index is defined as

$$P^x_n(\omega_1,\omega_2,\ldots,\omega_{n-1}) \triangleq \frac{C^x_n(\omega_1,\omega_2,\ldots,\omega_{n-1})}{\left[C^x_2(\omega_1)\cdot C^x_2(\omega_2)\cdots C^x_2(\omega_{n-1})\cdot C^x_2(\omega_1+\omega_2+\cdots+\omega_{n-1})\right]^{\frac{1}{2}}}. \qquad (2.52)$$

The third-order ($n=3$) coherency index is also called bicoherency (normalized bispectrum) [Hasselman et al., 1963; Raghuveer and Nikias, 1985]. The nth-order coherency index is very useful for the detection and characterization of non-linearities in time series via phase relations of their harmonic components. Also, the nth-order coherency index becomes useful in studying the phase response of non-Gaussian linear processes, i.e., processes whose spectra are modeled by the same linear filter. The magnitude of the nth-order coherency, $|P^x_n(\omega_1,\ldots,\omega_{n-1})|$, is called the coherence index.

2.3.10 Cross-Cumulant Spectra

The cross-cumulant spectra are defined as the multidimensional Fourier transforms of the corresponding cross-cumulants. Formation of the Fourier transform of the relationship (2.20) gives

$$C_{x_1x_2\ldots x_n}(\omega_1,\omega_2,\ldots,\omega_{n-1}) \triangleq \sum_{\tau_1=-\infty}^{+\infty}\cdots\sum_{\tau_{n-1}=-\infty}^{+\infty} c_{x_1\ldots x_n}(\tau_1,\tau_2,\ldots,\tau_{n-1})\exp\{-j(\omega_1\tau_1+\omega_2\tau_2+\cdots+\omega_{n-1}\tau_{n-1})\} \qquad (2.53)$$

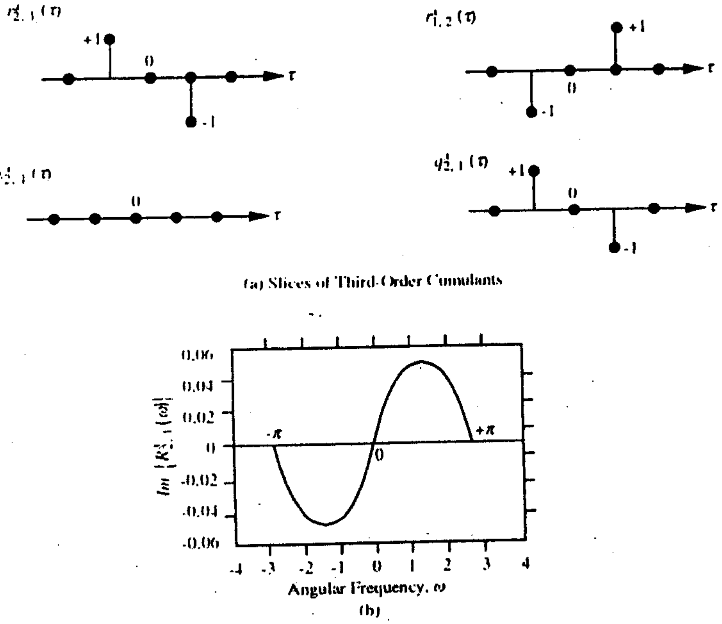


Figure 2.5 Random process $X(k) = W(k) - W(k-1)$ described in Example (2.6) (a) 1-d slices of its third-order cumulants, (b) the corresponding 1-d spectrum.

which is the nth-order cross-cumulant spectrum of processes $\{X_i(k)\}$, $i = 1, 2, \ldots, n$. The summability of the cross-cumulant sequence is assumed. For example, combining (2.18) and (2.53) we get a cross-bispectrum of $\{X(k)\}$ and $\{Y(k)\}$; i.e., [Brillinger, 1977]

$$C_{xyy}(\omega_1, \omega_2) = \sum_{\tau_1=-\infty}^{+\infty} \sum_{\tau_2=-\infty}^{+\infty} c_{xyy}(\tau_1, \tau_2) \exp\{-j(\omega_1\tau_1 + \omega_2\tau_2)\}. \qquad (2.54)$$

On the other hand, combining (2.17) and (2.53) we obtain

$$C_{xy}(\omega) = \sum_{\tau=-\infty}^{+\infty} c_{xy}(\tau) \exp\{-j\omega\tau\} \qquad (2.55)$$

which is the cross-spectrum (2nd-order) between $\{X(k)\}$ and $\{Y(k)\}$. Akaike [1966] defines as "mixed spectrum" the quantity

$$B_{xyy}(\omega; \sigma) \triangleq \cdot \sum_{\tau=-\infty}^{+\infty} c_{yyx}(\sigma, \tau) \exp\{-j\omega\tau\} \qquad (2.56)$$

as it relates to a spectral function mixed in time and frequency. The "mixed spectrum" $B_{xyy}(\omega; \sigma)$ gives the cross-spectrum between $\{X(k)\}$ and $\{Y(k)Y(k+\tau)\}$ under the assumption of being stationary. The cross-bispectrum $C_{yyx}(\omega_1, \omega_2)$ and the mixed spectrum $B_{yyx}(\omega; \sigma)$ satisfy the relationship [Akaike, 1966]

$$C_{yyx}(\omega_1, \omega_2) = \sum_{\sigma=-\infty}^{+\infty} B_{yyx}(\omega_2; \sigma) \exp\{-j\omega_q \sigma\}. \qquad (2.57)$$

Extensions of the "mixed spectrum" definition (2.56) to higher-order cases will be straightforward.

2.3.11 Linear Phase Shifts

Consider a zero-mean stationary random process $\{X(k)\}$ with finite moments up to order n. Let us form a new process $Y(k) = X(k-D)$ where D is a constant integer. From (2.5), (2.9), and (2.10) we conclude that

$$\begin{aligned} &\mathrm{Cum}[X(k), X(k+\tau_1), \ldots, X(k+\tau_{n-1})] = \\ &\mathrm{Cum}[Y(k), Y(k+\tau_1), \ldots, Y(k+\tau_{n-1})] \\ &= c_n^x(\tau_1, \tau_2, \ldots, \tau_{n-1}) \end{aligned} \qquad (2.58)$$

which implies that processes $\{X(k)\}$, $\{Y(k)\}$ have identical cumulant spectra. In other words, **cumulant spectra suppress linear phase shifts.**

On the other hand, if we form the signals $X_i(k) = X(k)$ for all $i \neq 2$, $X_2(k) = Y(k) = X(k-D)$, and generate the cross-cumulants

$$c_{xyx \ldots x}(\tau_1, \tau_2, \ldots, \tau_{n-1}) = \mathrm{Cum}[X(k), Y(k+\tau_1), X(k+\tau_2), \ldots, X(k+\tau_{n-1})]$$

we obtain

$$c_{xyx \ldots x}(\tau_1, \tau_2, \ldots, \tau_{n-1}) = c_n^x(\tau_1, \tau_2 - D, \tau_3, \ldots, \tau_{n-1}). \qquad (2.59)$$

Combining (2.53) and (2.59), we obtain

$$C_{xyx \ldots x}(\omega_1, \omega_2, \ldots, \omega_{n-1}) = C_n^x(\omega_1, \omega_2, \ldots, \omega_{n-1}) \cdot \exp\{j\omega_2 D\}. \qquad (2.60)$$

From (2.59) and (2.60), it is apparent that cross-cumulants and their corresponding cross-spectra do preserve linear phase shifts. Specifically, the cross-cumulant spectrum between $\{X(k)\}$ and $\{X(k-D)\}$ equals the cumulant spectrum of $\{X(k)\}$ times a linear phase shift component with slope determined by D. Extensions of (2.59) to higher-order cross-cumulant spectra are straightforward. This key property of cross-cumulants will be considered further later in this book in the study of time delay estimation and array processing problems (Chapter 8).

2.3.12 Complex Regression Coefficients

The complex "regression" coefficient, $R(\omega)$, of a process $\{X(k)\}$ on the process $\{Y(k)\}$, is defined by Brillinger [1977] as

$$R(\omega) \triangleq \frac{C_{xy}(\omega)}{C_{xx}(\omega)} \tag{2.61}$$

where $C_{xy}(\omega)$, $C_{xx}(\omega)$ are the cross-spectrum and power spectrum, respectively. Higher-order complex "regression" coefficients may be defined as

$$R_1(\omega_1,\omega_2) \triangleq \frac{C_{xxy}(\omega_1,\omega_2)}{C_{xy}(\omega_1)C_{xy}(\omega_2)} \tag{2.62}$$

and

$$R_2(\omega_1,\omega_2) \triangleq \frac{C_{yxx}(\omega_1,\omega_2)}{C_{xxx}(\omega_1,\omega_2)}. \tag{2.63}$$

Regression coefficients (2.61), (2.63) become useful, as we see later, in linear and non-linear system identification problems using cumulant spectra of input/output measurements [Rosenblatt, 1985].

2.3.13 Complex Processes

If a given process $\{X(k)\}$ is complex, its nth-order cumulant sequence has more than one definition depending on where we place the conjugation "$*$" operation. For example, 3rd-order cumulant sequences may be defined as

$$\begin{aligned} c_3^{(1)}(\tau_1,\tau_2) &\triangleq \text{Cum}[X(k),X(k+\tau_1),X(k+\tau_2)] \\ c_3^{(2)}(\tau_1,\tau_2) &\triangleq \text{Cum}[X(k),X^*(k+\tau_1),X(k+\tau_2)] \\ c_3^{(3)}(\tau_1,\tau_2) &\triangleq \text{Cum}[X(k),X^*(k+\tau_1),X^*(k+\tau_2)] \end{aligned} \tag{2.64}$$

etc., and their bispectra will follow from (2.30). Each one of these bispectrum functions will be different from the others. In general, there are 2^n different nth-order complex cumulant definitions.

Example 2.7

Let us consider the process

$$X(k) = a \exp\{j\omega_o k\}$$

where ω_o is constant and a is a random variable with $E\{a\} = 0$, $E\{a^2\} = Q$, $E\{a^3\} = 0$ and $E\{a^4\} = \mu$. Its second-order moments are given by (2.9).

$$\begin{aligned} \text{Mom}[X(k),X^*(k+\tau)] &= E\{X(k)X^*(k+\tau)\} \\ &= Q\exp\{j(-\omega_o\tau)\} \\ &= c_2^x(\tau), \end{aligned} \tag{2.65}$$

which implies that the process is wide-sense stationary.

Two of the sixteen possible definitions of the fourth-order moments of $\{X(k)\}$ are

$$\begin{aligned} \text{Mom}[X(k),X^*(k+\tau_1),X(k+\tau_2),X^*(k+\tau_3)] &= \mu\exp\{j\omega_o(\tau_2-\tau_1-\tau_3)\} \\ &= m_4^x(\tau_1,\tau_2,\tau_3) \end{aligned} \tag{2.66}$$

and

$$\begin{aligned} \text{Mom}\;&[X(k),X(k+\tau_1),X(k+\tau_2),X^*(k+\tau_3)] = \\ &= \mu\exp\{j(2\omega_o k)\}\cdot\exp\{j\omega_o(\tau_2+\tau_1-\tau_3)\} = \\ &m_4^x(k;\tau_1,\tau_2,\tau_3). \end{aligned}$$

Here we see two different fourth-order moment sequences of the process where only the first one is stationary. Combining (2.65) and (2.66) with (2.14), we obtain

$$c_4^x(\tau_1,\tau_2,\tau_3) = (\mu-3Q^2)\exp\{j\omega_o(\tau_2-\tau_1-\tau_3)\}. \tag{2.67}$$

Combining (2.46), (2.47), and (2.67), we can get the following $1-d$ slices of $c_4^x(\tau_1,\tau_2,\tau_3)$:

$$\begin{aligned} r_{3,1}^x(\tau) &= \gamma\exp\{j(-\omega_o\tau)\} \\ r_{1,3}^x(\tau) &= \gamma\exp\{j(-\omega_o\tau)\} \\ r_{2,2}^x(\tau) &= \gamma \end{aligned} \tag{2.68}$$

where $\gamma = \mu - 3Q^2$. If we substitute (2.68) into (2.47), we obtain

$$\begin{aligned} s^x_{3,1}(\tau) &= \gamma \exp\{j(-\omega_o\tau)\} \\ q^x_{3,1}(\tau) &= 0 \\ s^x_{2,2}(\tau) &= \gamma. \end{aligned}$$

Substituting these results into (2.48), we see that

$$R^x_{3,1}(\omega) = \frac{\gamma}{Q} \cdot C^x_2(\omega)$$

where $C^x_2(\omega)$ is the power spectrum of the process. □

2.3.14 The Wigner Bispectrum

Gerr [1988] introduced the third-order Wigner time-frequency distribution or Wigner bispectrum which is a mixed time-frequency representation that extends the standard Wigner distribution in the same way that the bispectrum extends the power spectrum. The Wigner time-frequency distribution of a real-valued signal $\{X(k)\}$ may be defined as

$$W_2(t,\omega) = \int X(t+a(\tau))X(t+b(\tau)) \exp\{-j\omega\tau\}d\tau \tag{2.69}$$

where $a(\tau) \triangleq \tau/2$, $b(\tau) \triangleq \tau/2$. If $\{X(t)\}$ is a zero-mean stationary random process, then

$$\begin{aligned} E\{W_2(t,\omega)\} &= \int c^x_2(\tau) \exp\{-j\omega\tau\}d\tau \\ &= C^x_2(\omega) \end{aligned} \tag{2.70}$$

where $c^x_2(\tau)$ is the covariance function and $C^x_2(\omega)$ is the power spectral density of $\{X(t)\}$. Analogous to (2.69), the Wigner bispectrum is defined as [Gerr, 1988]

$$\begin{aligned} W_3(t,\omega_1,\omega_2) \triangleq & \iint X(t+a(\tau_1,\tau_2))X(t+b(\tau_1,\tau_2))X(t+c(\tau_1,\tau_2)) \\ & \cdot \exp\{-j(\omega_1+\omega_2)\tau_1 - j\omega_2\tau_2\}d\tau_1 d\tau_2 \end{aligned} \tag{2.71}$$

where the lag functions are given by

$$\begin{aligned} a(\tau_1,\tau_2) &= -\frac{2}{3}\tau_1 - \frac{1}{3}\tau_2 \\ b(\tau_1,\tau_2) &= \frac{1}{3}\tau_1 - \frac{1}{3}\tau_2 \\ c(\tau_1,\tau_2) &= \frac{1}{3}\tau_1 + \frac{2}{3}\tau_2. \end{aligned}$$

It is easy to see from (2.71) that

$$\begin{aligned} E\{W_3(t,\omega_1,\omega_2)\} &= \iint c^x_3(\tau_1, \tau_1+\tau_2) \cdot \exp\{-j(\omega_1+\omega_2)\tau_1 - j\omega_2\tau_2\}d\tau_1 d\tau_2 \\ &= C^x_3(\omega_1,\omega_2) \end{aligned} \tag{2.72}$$

where $c^x_3(\tau_1,\tau_2)$, $C^x_3(\omega_1,\omega_2)$ are the third-order cumulants and bispectrum of $\{X(t)\}$, respectively.

The Wigner bispectrum may prove useful for extracting time-varying phase information or phase coupling between frequency components, as well as for detection, parameter estimation, and classification of deterministic signals in stochastic noise.

2.4 CUMULANT SPECTRA OF NON-GAUSSIAN LINEAR PROCESSES

Let $\{X(k)\}$ be a zero-mean non-Gaussian process with all its moments finite up to order nth (stationary to order n) and with cumulant spectrum $C^x_n(\omega_1,\omega_2,\ldots,\omega_{n-1})$. Assume that $\{X(k)\}$ is the input to a linear time-invariant (LTI) system described by

$$Y(k) = \sum_{i=-\infty}^{+\infty} h(k-i)\,X(i) \tag{2.73}$$

where

$$h(k) \triangleq \frac{1}{2\pi}\int_{-\pi}^{+\pi} H(\omega)\exp\{j\omega k\}d\omega \tag{2.74}$$

is the impulse response of the system and $H(\omega)$ its frequency response function. The LTI system is assumed to be stable; i.e., its impulse response is absolutely summable

$$\sum_{k=-\infty}^{+\infty} |h(k)| < \infty.$$

2.4.1 Cumulant Spectra of LTI Systems

Brillinger and Rosenblatt [1967 a and b] established that the nth-order cumulant spectra of the input $\{X(k)\}$ and output $\{Y(k)\}$ are related by

$$\begin{aligned} C^y_n(\omega_1,\ldots,\omega_{n-1}) = H(\omega_1)\cdot H(\omega_2)\cdots H(\omega_{n-1}) &\cdot H^*(\omega_1+\cdots+\omega_{n-1}) \\ &\cdot C^x_n(\omega_1,\ldots,\omega_{n-1}). \end{aligned} \tag{2.75}$$