School of Mechanical Engineering

Sharif University of Technology

Convection Heat Transfer

Instructor: M.H.Saidi

PhD, Professor

Lecture #4

Fall 2011

Boundary Layer Flow

Summary of second session:

- *Scale Analysis*
- *Introduction to boundary layer flow*

- This session: Laminar Boundary Layer Flow(Velocity and Temperature fields)

Convection Heat Transfer

LAMINAR BOUNDARY LAYER FLOW

CONCEPT OF BOUNDARY LAYER

In the **d** × L region, then, the longitudinal momentum equation (8) accounts for the competition between three types of forces:

Inertia	Pressure	Friction
$U_\infty \dfrac{U_\infty}{L}, \quad v \dfrac{U_\infty}{\delta}$	$\dfrac{P}{\rho L}$	$\nu \dfrac{U_\infty}{L^2}, \quad \nu \dfrac{U_\infty}{\delta^2}$

(14)

From the mass continuity equation, we have:

$$\frac{U_\infty}{L} \sim \frac{v}{\delta}$$

(15)

LAMINAR BOUNDARY LAYER FLOW

CONCEPT OF BOUNDARY LAYER

If the boundary layer region **d** × L is slender(not Bluff), such that:

$$\delta \ll L \qquad (16)$$

The last scale in Eq. (14) is the scale most representative of the friction force. Neglecting the term $\partial^2 U / \partial x^2$ at the expense of the $\partial^2 U / \partial y^2$ term in the x momentum Equation (8) yields:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \qquad (17)$$

Similarly the y momentum equation reduces to:

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \frac{\partial^2 v}{\partial y^2} \qquad (18)$$

LAMINAR BOUNDARY LAYER FLOW

CONCEPT OF BOUNDARY LAYER

Some treatment on Eqs.17 & 18:

The pressure at any point in the fluid is a function of both x and y:

$$dP = \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy \qquad (19)$$

$$\frac{dP}{dx} = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y}\frac{dy}{dx} \qquad (20)$$

The orders of magnitude of the two pressure gradients can be deduced by recognizing a balance between pressure forces and either friction or inertia .

For instance, the pressure ~ friction balance **in Eqs. (17) & (18) suggest that :**

$$\frac{\partial P}{\partial x} \sim \frac{\mu U_\infty}{\delta^2} \qquad (21)$$

$$\frac{\partial P}{\partial y} \sim \frac{\mu v}{\delta^2} \qquad (22)$$

LAMINAR BOUNDARY LAYER FLOW

CONCEPT OF BOUNDARY LAYER

Some treatment on Eqs.17 & 18:

Turning our attention to the right-hand side of Eq. (20) and the ratio of two terms:

$$\frac{dP}{dx} = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y}\frac{dy}{dx} \qquad (20)$$

$$\frac{(\partial P/\partial y)(dy/dx)}{\partial P/\partial x} \sim \frac{v\delta}{U_\infty L} \sim \left(\frac{\delta}{L}\right)^2 \ll 1 \qquad (23)$$

$$\frac{dP}{dx} = \frac{\partial P}{\partial x} \qquad (24)$$

This means that in the boundary layer, *the pressure varies mainly in the longitudinal direction. Or* the pressure at any point inside the boundary layer region is practically the same as the pressure immediately outside it.

$$\frac{\partial P}{\partial x} = \frac{dP_\infty}{dx} \qquad (25)$$

LAMINAR BOUNDARY LAYER FLOW

CONCEPT OF BOUNDARY LAYER

Some treatment on Eqs.17 & 18:

Making this substitution in x momentum equation (17), we obtain:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (26)$$

it is a statement of momentum conservation in both the x and y directions.
The boundary layer equation for *energy follows* from Eq. (10), neglecting the term accounting for thermal diffusion in the x direction:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \qquad (27)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (7)$$

Toghether with mass conservation equation, we have three eqautions and three unknowns: {u,v,T}

LAMINAR BOUNDARY LAYER FLOW

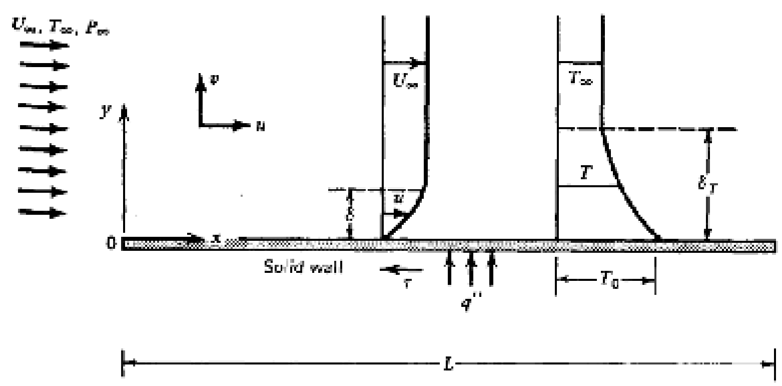
VELOCITY AND THERMAL BOUNDARY LAYERS

Let δ be the thickness of the region in which u varies from 0 at the wall to U_∞ in the free stream. Let δ_T be the thickness of another slender region in which T varies from T_0 at the wall to T_∞ in the free stream.

Keeping up with tradition, in the present treatment we refer to δ and δ_T as the velocity boundary layer thickness and the thermal boundary layer thickness, respectively.

In scaling terms, the flow friction term will be :

$$\tau \sim \mu \frac{U_\infty}{\delta} \qquad (28)$$



LAMINAR BOUNDARY LAYER FLOW

VELOCITY AND THERMAL BOUNDARY LAYERS

❖**Scale analysis implies that in order to estimate the wall shear stress one should evaluate the extent friction δ.**

❖**Considering the flow over a flat horizontal plate, in which:** $dP_\infty/dx = 0$

❖**The boundary layer momentum equation implies that:**

$$\text{inertia} \sim \text{friction}$$

$$\frac{U_\infty^2}{L}, \frac{\upsilon U_\infty}{\delta} \sim \nu \frac{U_\infty}{\delta^2} \qquad (29)$$

❖**Referring to the mass continuity scaling (15):** $\frac{U_\infty}{L} \sim \frac{\upsilon}{\delta}$ **, we conclude that**

❖***the two inertia terms are of the same order of magnitude*. As such from (29), we have:**

$$\delta \sim \left(\frac{\nu L}{U_\infty}\right)^{1/2} \qquad (30)$$

In other words:

$$\frac{\delta}{L} \sim \mathrm{Re}_L^{-1/2} \qquad (31)$$

LAMINAR BOUNDARY LAYER FLOW

VELOCITY AND THERMAL BOUNDARY LAYERS

- **Equation (31) states that the slenderness postulate (δ « L) is valid provided that:**

$$Re^{1/2}{}_L >> 1$$

- **Which can be used to assess the limitations of the boundary layer analysis: For example, the boundary layer solution will fail in the tip region of length**

- **The wall shear stress scales as:**

$$\tau \sim \mu \frac{U_\infty}{L} Re_L^{1/2} \sim \rho U_\infty^2 Re_L^{-1/2} \quad (32)$$

- **The dimensionless skin friction coefficient** $C_f = t/(rv^2/2)$ **depends on the Reynolds number,**

$$C_f \sim Re_L^{-1/2} \quad (33)$$

LAMINAR BOUNDARY LAYER FLOW

VELOCITY AND THERMAL BOUNDARY LAYERS

- **The heat transfer engineering question [Eq. (6)] is answered by focusing**
- **on the thermal boundary layer of thickness δ_T:**

$$h = \frac{-k(\partial T/\partial y)_{y=0}}{T_0 - T_\infty} \rightarrow h \sim \frac{k(\Delta T/\delta_T)}{\Delta T} \sim \frac{k}{\delta_T} \qquad (34)$$

- **Where:** $\Delta T = T_0 - T_\infty$

- `
- **From Eq.(27):** $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$ **,we *can write a balance between conduction from the wall into the stream and convection (enthalpy flow) parallel to the wall:***

$$\text{convection} \sim \text{conduction}$$

$$u\frac{\Delta T}{L}, v\frac{\Delta T}{\delta_T} \sim \alpha\frac{\Delta T}{\delta_T^2} \qquad (35)$$

The δ_T scale needed for estimating h can be determined analytically in the two limits discussed in the next slide.

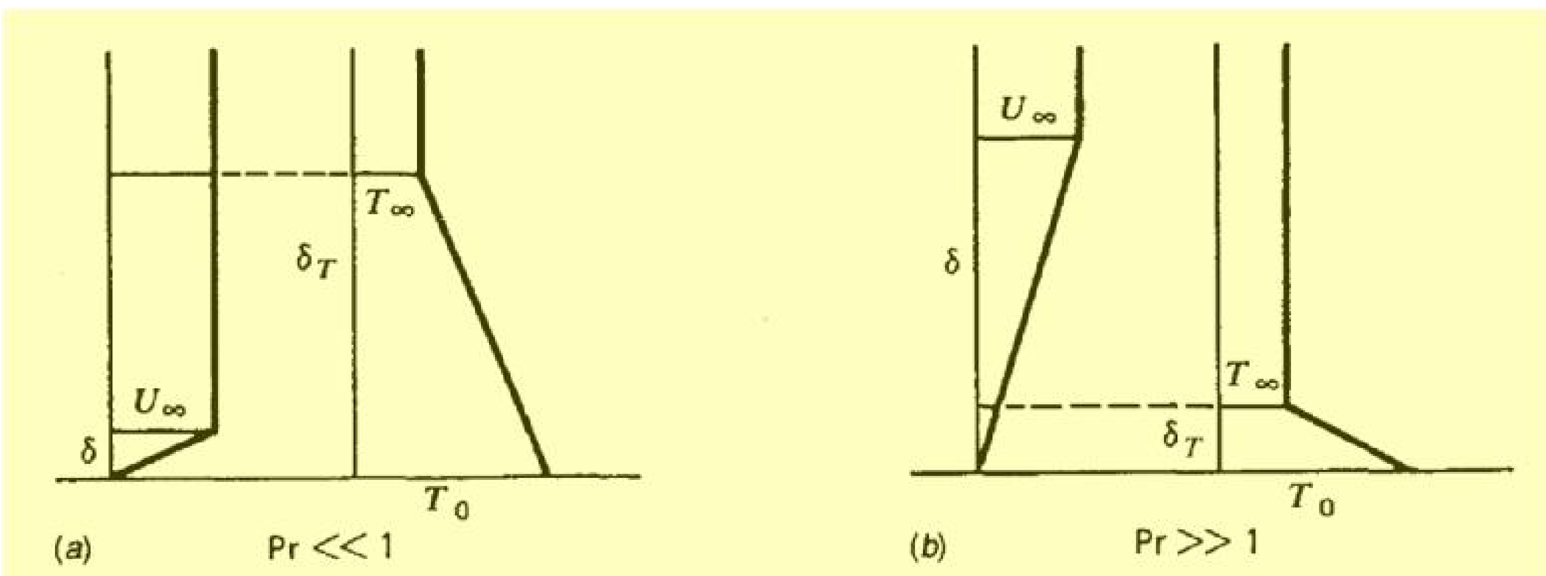
Convection Heat Transfer

LAMINAR BOUNDARY LAYER FLOW

VELOCITY AND THERMAL BOUNDARY LAYERS

Two limits for estimating δ_T

1- Thick thermal boundary layer, $d_T >> d$
2- Thin thermal boundary layer, $d_T << d$



Prandtl number effect on the relative thickness of the velocity and temperature boundary layers

Convection Heat Transfer

LAMINAR BOUNDARY LAYER FLOW

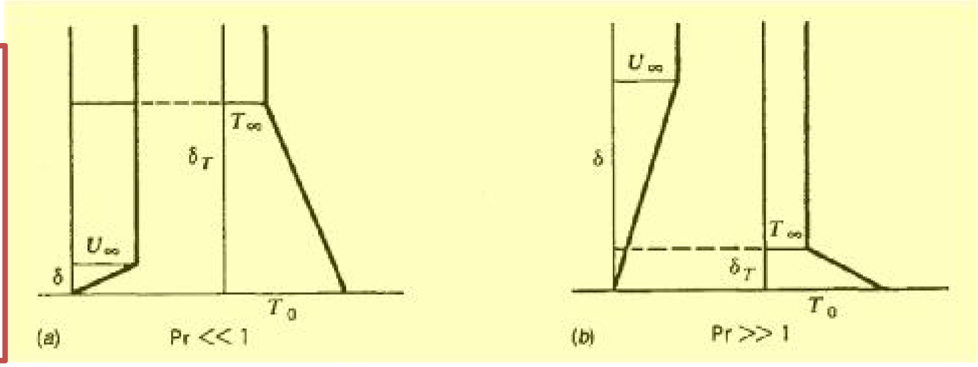
VELOCITY AND THERMAL BOUNDARY LAYERS

1- Thick thermal boundary layer, $d_T >> d$

The u scale outside the velocity boundary layer (and inside the thermal layer) is U∞

According to Eq. (15): $\frac{U_\infty}{L} \sim \frac{v}{\delta}$

the v scale in the same region is $v \sim U_\infty \delta / L$



convection ~ conduction

$$u \frac{\Delta T}{L}, v \frac{\Delta T}{\delta_T} \sim \alpha \frac{\Delta T}{\delta_T^2}$$

$$v \frac{\Delta T}{\delta_T} \sim U_\infty \frac{\Delta T}{L} \frac{\delta}{\delta_T}$$

$$U_\infty \frac{\Delta T}{L}$$

$$\frac{\delta}{\delta_T} << 1$$

(36)

$$u \frac{\Delta T}{L} >> v \frac{\Delta T}{\delta_T}$$

LAMINAR BOUNDARY LAYER FLOW

VELOCITY AND THERMAL BOUNDARY LAYERS

1- Thick thermal boundary layer, $\delta_T >> \delta$

In conclusion, the convection ~ conduction balance expressed by the energy equation (35) is simply $(U_\infty \, \Delta T)/L \sim (\alpha \, \Delta T)/\delta_T^2$ **which yields:**

$$\frac{\delta_T}{L} \sim \mathrm{Pe}_L^{-1/2} \sim \mathrm{Pr}^{-1/2}\, \mathrm{Re}_L^{-1/2} \qquad (37)$$

where $\mathrm{Pe}_L = U_\infty L/\alpha$

Comparing with (21): $\frac{\delta}{L} \sim \mathrm{Re}_L^{-1/2}$

We find that the relative size of δ_T and δ depends on the *Prandtl number*

$\mathrm{Pr} = \nu/\alpha$

$$\frac{\delta_T}{\delta} \sim \mathrm{Pr}^{-1/2} >> 1 \qquad (38)$$

LAMINAR BOUNDARY LAYER FLOW

VELOCITY AND THERMAL BOUNDARY LAYERS

1- Thick thermal boundary layer, $d_T >> d$ (Pr = .01(liquid metals)

$$\frac{\delta_T}{\delta} \sim \mathrm{Pr}^{-1/2} >> 1$$

The first assumption, $d_T >> d$, is therefore valid in the $Pr^{1/2} << 1$
Which represents the range occupied by liquid metals.

The heat transfer coefficient corresponding to the low-Prandtl number limit is:

$$h \sim \frac{k}{L} \mathrm{Pr}^{1/2} \mathrm{Re}_L^{1/2} \qquad (\mathrm{Pr} \ll 1) \qquad (39)$$

or, expressed as a Nusselt number $\mathrm{Nu} = hL/k$

$$\mathrm{Nu} \sim \mathrm{Pr}^{1/2} \mathrm{Re}_L^{1/2} \qquad (40)$$

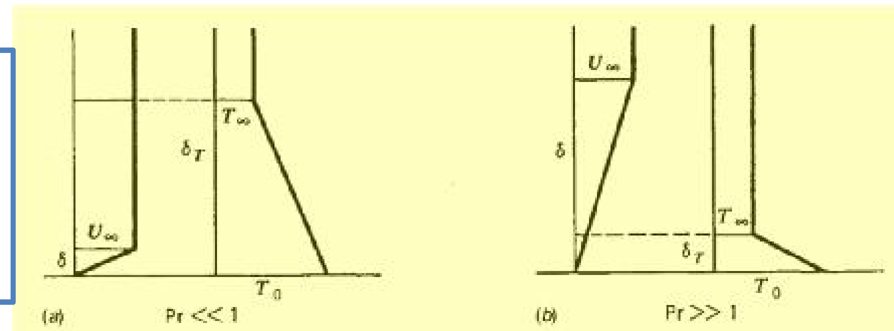
Convection Heat Transfer

LAMINAR BOUNDARY LAYER FLOW

VELOCITY AND THERMAL BOUNDARY LAYERS

2- Thin thermal boundary layer, $d_T << d$ (Pr= 100, Oils)

considerable interest is the class of fluids with Prandtl numbers of order unity (e.g., air) or greater than unity (e.g., water or oils).



$$\text{convection} \sim \text{conduction}$$

$$u\frac{\Delta T}{L},\ v\frac{\Delta T}{\delta_T} \sim \alpha\frac{\Delta T}{\delta_T^2}$$

$$u \sim U_\infty \frac{\delta_T}{\delta} \qquad (41)$$

$$\frac{\delta_T}{L} \sim \mathrm{Pr}^{-1/3}\,\mathrm{Re}_L^{-1/2} \qquad (42)$$

which means that :

$$\frac{\delta_T}{\delta} \sim \mathrm{Pr}^{-1/3} << 1 \qquad (43)$$

LAMINAR BOUNDARY LAYER FLOW

VELOCITY AND THERMAL BOUNDARY LAYERS

2- Thin thermal boundary layer, $d_T << d$

$$\frac{\delta_T}{\delta} \sim \mathrm{Pr}^{-1/3} \ll 1$$

The first assumption, $d_T << d$ **, is therefore valid in the** $Pr^{1/3} >> 1$ **fluids.**
Which represents the range occupied by oils.

The heat transfer coefficient corresponding to the high-Prandtl number limit is:

$$h \sim \frac{k}{L} \mathrm{Pr}^{1/3} \mathrm{Re}_L^{1/2} \qquad (\mathrm{Pr} \gg 1) \qquad (44)$$

or, expressed as a Nusselt number

$$\mathrm{Nu} \sim \mathrm{Pr}^{1/3} \mathrm{Re}_L^{1/2} \qquad (\mathrm{Pr} \gg 1) \qquad (45)$$

LAMINAR BOUNDARY LAYER FLOW

Concept of Reynolds number

Referring to Eq.(31): $\frac{\delta}{L} \sim Re_L^{-1/2}$ **which is the first place we encounter the Reynolds number in external flow,** $Re_L = U_\infty L/\nu$.

Basically, the Reynolds number is described as the order of magnitude of the inertia/friction ratio in a particular flow. This interpretation is not always correct because at least in the boundary layer region examined above, there is always a balance between inertia and friction, whereas Re can reach as high as 10^5 before the transition to turbulent flow.

The only physical interpretation of the Reynolds number in boundary layer flow appears to be:

$$Re_L^{1/2} = \frac{\text{wall length}}{\text{boundary layer thickness}}$$

Next session:

- *Laminar Boundary Layer Flow:*
  Integral Solution