School of Mechanical Engineering

Sharif University of Technology

Convection Heat Transfer

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Lecture #5

Fall 2011

Boundary Layer Flow

Summary of previous session

- ***Laminar Boundary Layer Flow:***

Flow(Velocity and Temperature fields)

- This session:

Laminar Boundary Layer Flow:

Integral Solution

Convection Heat Transfer

LAMINAR BOUNDARY LAYER FLOW

INTEGRAL SOLUTIONS

The next step in the sequence of refining the answers to the friction and heat transfer questions amounts to determining the numerical coefficients (factors) missing from the scaling laws.

In the realm of scale analysis, we made no distinction between the local values of r and h (the values right at x = L) and the average values τ_{0-L} and h_{0-L} defined as:

$$\tau_{0-L} = \frac{1}{L}\int_0^L \tau\, dx, \qquad h_{0-L} = \frac{1}{L}\int_0^L h\, dx \qquad (46)$$

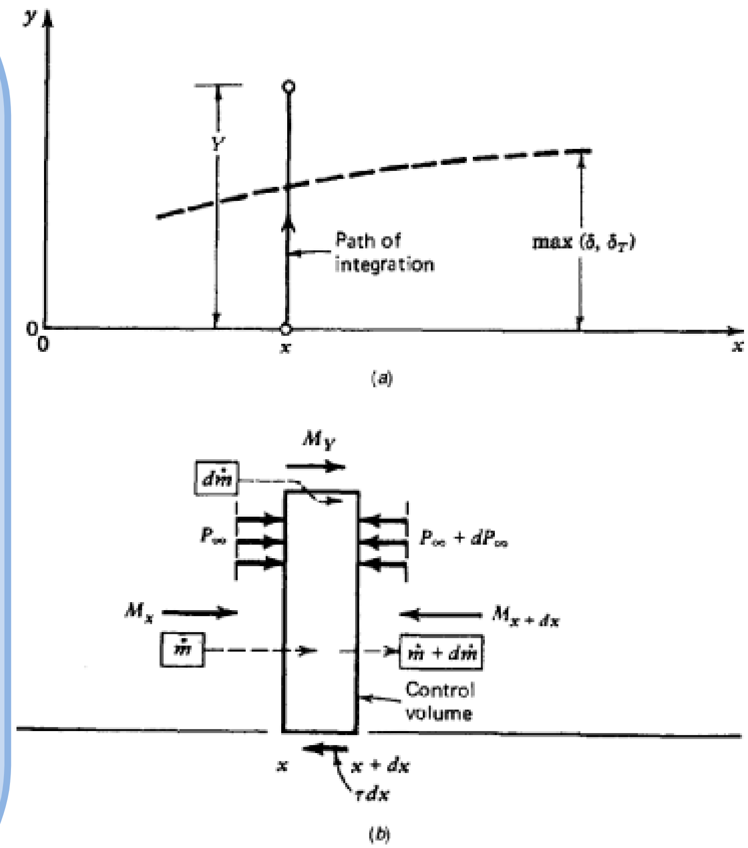
LAMINAR BOUNDARY LAYER FLOW

INTEGRAL SOLUTIONS

- In the integral method, we recognize that what we need is not a complete solution for the velocity u(x,y) and temperature T{x,y) near the wall, but only the gradients $\partial(u,T)/\partial y$ evaluated at y = 0.

- We have the opportunity to simplify the boundary layer equations by eliminating y as a variable.

- This is accomplished by integrating each equation term by term from y = 0 to y = Y, where Y > max (δ, δ_T) is situated in the free stream.



Convection Heat Transfer

LAMINAR BOUNDARY LAYER FLOW

INTEGRAL SOLUTIONS

u +

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (7)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (26)$$

$$\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) = -\frac{1}{\rho}\frac{dP_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

(47)

T ´ +

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (7)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \qquad (27)$$

$$\frac{\partial}{\partial x}(uT) + \frac{\partial}{\partial y}(vT) = \alpha\frac{\partial^2 T}{\partial y^2}$$

(48)

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$$\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) = -\frac{1}{\rho}\frac{dP_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (47)$$

$$\frac{\partial}{\partial x}(uT) + \frac{\partial}{\partial y}(vT) = \alpha\frac{\partial^2 T}{\partial y^2} \qquad (48)$$

Integrating Eqs. (47) and (48) from y = 0 to y = Y, and using Leibnitz's integral formula, yields :

$$\frac{d}{dx}\int_0^Y u^2\, dy + u_Y v_Y - u_0 v_0 = -\frac{1}{\rho} Y \frac{dP_\infty}{dx} + \nu\left(\frac{\partial u}{\partial y}\right)_Y - \nu\left(\frac{\partial u}{\partial y}\right)_0 \qquad (49)$$

$$\frac{d}{dx}\int_0^Y uT\, dy + v_Y T_Y - v_0 T_0 = \alpha\left(\frac{\partial T}{\partial y}\right)_Y - \alpha\left(\frac{\partial T}{\partial y}\right)_0 \qquad (50)$$

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INTEGRAL SOLUTIONS

Since the free stream is uniform, we note that $(\partial/\partial y)_Y = 0$, $U_Y = U_\infty$, and $T_Y = T_\infty$. Also, since the wall is impermeable, $v_0 = 0$, and v_Y by performing the same integral on the continuity Equation (7):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Longrightarrow \quad \frac{d}{dx}\int_0^Y u\, dy + v_Y - v_0 = 0 \qquad (51)$$

Substituting Vy into Eqs. (49) and (50), assuming that T_∞ is, in general, a function of x and rearranging the resulting expression, we obtain:

$$\frac{d}{dx}\int_0^Y u(U_\infty - u)\, dy = \frac{1}{\rho} Y \frac{dP_\infty}{dx} + \frac{dU_\infty}{dx}\int_0^Y u\, dy + \nu\left(\frac{\partial u}{\partial y}\right)_0 \qquad (52)$$

$$\frac{d}{dx}\int_0^Y u(T_\infty - T)\, dy = \frac{dT_\infty}{dx}\int_0^Y u\, dy + \alpha\left(\frac{\partial T}{\partial y}\right)_0 \qquad (53)$$

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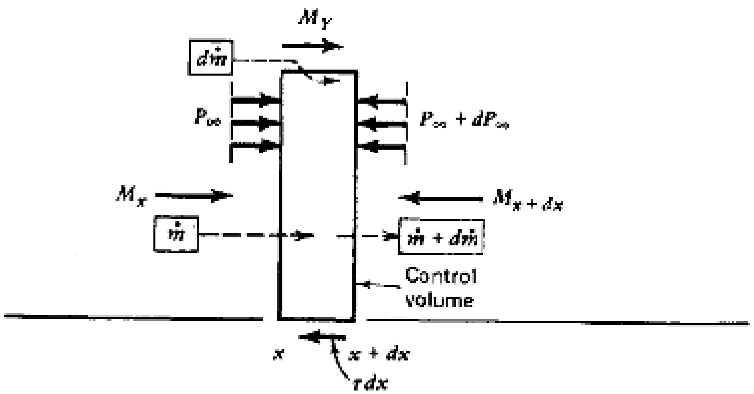
$$\frac{d}{dx}\int_0^Y u(U_\infty - u)\,dy = \frac{1}{\rho}Y\frac{dP_\infty}{dx} + \frac{dU_\infty}{dx}\int_0^Y u\,dy + \nu\left(\frac{\partial u}{\partial y}\right)_0$$

$$\frac{d}{dx}\int_0^Y u(T_\infty - T)\,dy = \frac{dT_\infty}{dx}\int_0^Y u\,dy + \alpha\left(\frac{\partial T}{\partial y}\right)_0$$

Integral boundary layer equations for momentum and energy

They account for the conservation of momentum and energy not at every point (x,y) as Eqs. (26) and (27), but in every slice of thickness dx and height Y



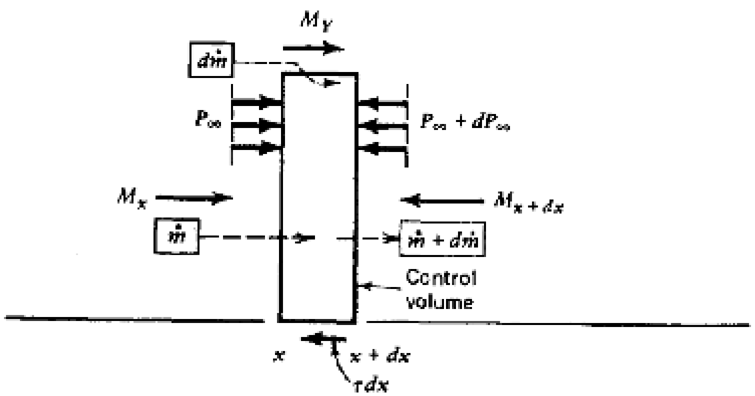
LAMINAR BOUNDARY LAYER FLOW

INTEGRAL SOLUTIONS

- **Equations (52) and (53) can also be derived by invoking the x momentum theorem and the first law of thermodynamics .**

- **For example, the momentum Equation (52) represents the following force balance:**

Forces acting from left to right on the control volume



$M_x = \int_0^Y \rho u^2\, dy$	Impulse due to the flow of a stream *into* the control volume
$M_Y = U_\infty\, d\dot{m}$	Impulse due to the flow of fast fluid (U_∞) into the control volume, at a rate $d\dot{m}$, where $\dot{m} = \int_0^Y \rho u\, dy$ is the mass flow rate through the slice of height Y
$P_\infty Y$	Pressure force

Convection Heat Transfer

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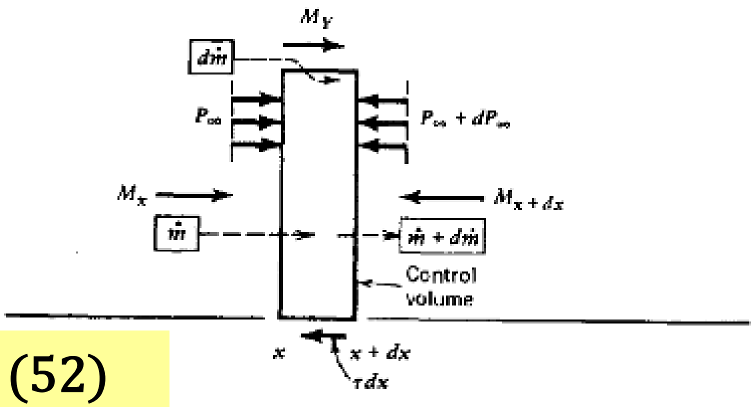
INTEGRAL SOLUTIONS

Forces acting from right to left on the control volume

$M_{x+dx} = M_x + (dM_x/dx)\,dx$	Reaction force due to flow of a stream *out* of the control volume
$\tau\,dx$	Tangential force due to friction
$Y[P_\infty + (dP_\infty/dx)\,dx]$	Pressure force

Setting the resultant of all these forces equal to zero, we derive Eq. (52).



$$\frac{d}{dx}\int_0^Y u(U_\infty - u)\,dy = \frac{1}{\rho} Y \frac{dP_\infty}{dx} + \frac{dU_\infty}{dx}\int_0^Y u\,dy + \nu\left(\frac{\partial u}{\partial y}\right)_0 \qquad (52)$$

INTEGRAL SOLUTIONS

- Consider the simplest laminar boundary layer problem—the uniform flow (U_∞ , P_∞= constants).

- To solve for the wall shear stress, let us assume that the shape of the longitudinal velocity profile is described by:

$$u = \begin{cases} U_\infty m(n), & 0 \le n \le 1 \\ U_\infty, & 1 \le n \end{cases} \qquad (54)$$

LAMINAR BOUNDARY LAYER FLOW

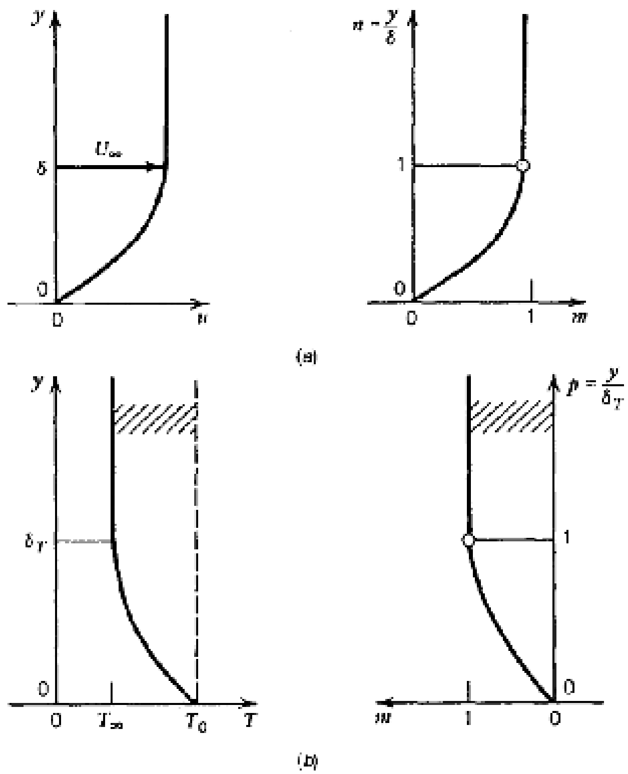
INTEGRAL SOLUTIONS

where m is an unspecified shape function that varies from 0 to 1 and $n = y/\delta$

$$\frac{d}{dx}\int_0^Y u(U_\infty - u)\,dy = \frac{1}{\rho}Y\frac{dP_\infty}{dx} + \frac{dU_\infty}{dx}\int_0^Y u\,dy + \nu\left(\frac{\partial u}{\partial y}\right)_0 \quad (52)$$

Substituting this assumption into Eq. (52) and noting that

$dP_\infty/dx = 0$ and $dU_\infty/dx = 0$ yields a first-order ordinary differential equation for the velocity boundary layer thickness $\delta(x)$



Selection of (a) velocity profile and (b) temperature profile for integral boundary layer analysis.

INTEGRAL SOLUTIONS

- **After substitution, we have:**

$$\delta \frac{d\delta}{dx}\left[\int_0^1 m(1-m)\,dn\right] = \frac{\nu}{U_\infty}\left(\frac{dm}{dn}\right)_{n=0} \quad (55)$$

- **The resulting expressions for local boundary layer thickness and skin friction coefficient are :**

$$\frac{\delta}{x} = a_1\,\mathrm{Re}_x^{-1/2} \quad (56)$$

$$C_{f,x} = \frac{\tau}{\frac{1}{2}\rho U_\infty^2} = a_2\,\mathrm{Re}_x^{-1/2} \quad (57)$$

- **Where:**

$$a_1 = \left[\frac{2(dm/dn)_{n=0}}{\int_0^1 m(1-m)\,dn}\right]^{1/2} \quad (56)'$$

$$a_2 = \left[2\left(\frac{dm}{dn}\right)_{n=0}\int_0^1 m(1-m)\,dn\right]^{1/2} \quad (57)'$$

Convection Heat Transfer

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INTEGRAL SOLUTIONS

The numerical coefficients a_1 and a_2 depend on the assumption made for the profile shape function m: Table shows that as long as this shape is reasonable, the choice of m(n) does not influence the skin friction result appreciably.

Profile Shape $m(n)$ or $m(p)$ (Fig. 2.4)	$\frac{\delta}{x}Re_x^{1/2}$	$\frac{C_{f,x}}{Re_x^{1/2}}$	$Nu\,Re_x^{-1/2}\,Pr^{-1/3}$ Uniform Temperature (Pr > 1)	$Nu\,Re_x^{-1/2}\,Pr^{-1/3}$ Uniform Heat Flux (Pr > 1)
$m = n$	3.46	0.577	0.289	0.364
$m = (n/2)(3 - n^2)$	4.64	0.646	0.331	0.417
$m = \sin(\pi n/2)$	4.8	0.654	0.337	0.424
Similarity solution	4.92[a]	0.664	0.332	0.453

[a]Thickness defined as the y value corresponding to $u/U_\infty = 0.99$.

LAMINAR BOUNDARY LAYER FLOW

INTEGRAL SOLUTIONS

Heat transfer coefficient information is extracted in a similar method from Eq. (53) with $dT/dx = 0$. Thus, we assume the temperature profile shapes:

$$T_0 - T = (T_0 - T_\infty)m(p), \qquad 0 \le p \le 1$$
$$T = T_\infty, \qquad 1 \le p \tag{58}$$

where $p = y/\delta_T$. We assume that :

$$\frac{\delta_T}{\delta} = \Delta \tag{59}$$

where Δ is a function of Prandtl number only and δ is given by Eq. (56): $\frac{\delta}{x} = a_1 \, \mathrm{Re}_x^{-1/2}$
Based on these assumptions and $\delta_T < \delta$ (high-Pr fluids), the integral energy Equation (53) $\frac{d}{dx}\int_0^Y u(T_\infty - T)\,dy = \frac{dT_\infty}{dx}\int_0^Y u\,dy + \alpha\left(\frac{\partial T}{\partial y}\right)_0$ reduces to:

$$\mathrm{Pr} = \frac{2(dm/dp)_{p=0}}{(a_1\Delta)^2}\left[\int_0^1 m(p\Delta)[1 - m(p)]\,dp\right]^{-1} \tag{60}$$

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Assuming the simplest temperature profile, m = p, Eq.(60) becomes:

$$\Delta = \mathrm{Pr}^{-1/3} \qquad (61)$$

The results usually listed in the literature correspond to the cubic profile:

$$m = (p/2)(3 - p^2)$$

$$\Delta = \frac{\delta_T}{\delta} = 0.976\mathrm{Pr}^{-1/3} \qquad (62)$$

$$h = 0.331\,\frac{k}{x}\,\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2} \qquad (63)$$

$$\mathrm{Nu} = \frac{hx}{k} = 0.331\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2} \qquad (64)$$

The local heat transfer results listed above are anticipated correctly by the scale Analysis Eqs.(44) and (45):

$$h \sim \frac{k}{L}\,\mathrm{Pr}^{1/3}\,\mathrm{Re}_L^{1/2} \qquad (\mathrm{Pr} \gg 1)$$

$$\mathrm{Nu} \sim \mathrm{Pr}^{1/3}\,\mathrm{Re}_L^{1/2} \qquad (\mathrm{Pr} \gg 1)$$

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INTEGRAL SOLUTIONS

In the case of liquid metals ($\Delta >> 1$), instead of Eq. (60), we obtain;

$$\mathrm{Pr} = \frac{2(dm/dp)_{p=0}}{(a_1\Delta)^2}\left[\int_0^{1/\Delta} m(p\Delta)[1-m(p)]\,dp + \int_{1/\Delta}^{1}[1-m(p)]\,dp\right]^{-1} \quad (65)$$

The sum of two integrals stems from the fact that when $\delta_T >> \delta$, immediately next to the wall ($0 < y < \delta$), the velocity is described by the assumed shape cm, whereas for $\delta < y < \delta_T$, the velocity is uniform, $u = \delta_T$ [Eq. (54)].
Since Δ is much greater than unity, the second integral dominates in Eq. (65).
Taking again the simplest profile $m = p$, we obtain:

$$\Delta = \frac{\delta_T}{\delta} = (3\mathrm{Pr})^{-1/2} \qquad (\mathrm{Pr} \ll 1) \quad (66)$$

$$\frac{\delta_T}{x} = 2\mathrm{Pr}^{-1/2}\,\mathrm{Re}_x^{-1/2} \qquad (\mathrm{Pr} \ll 1) \quad (67)$$

Convection Heat Transfer

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we derive the local heat transfer coefficient:

$$h = \frac{k}{\delta_T} = \frac{1}{2}\frac{k}{x}\,\mathrm{Pr}^{1/2}\,\mathrm{Re}_x^{1/2} \qquad (\mathrm{Pr} \ll 1) \tag{68}$$

or the local Nusselt number

$$\mathrm{Nu} = \frac{hx}{k} = \frac{1}{2}\,\mathrm{Pr}^{1/2}\,\mathrm{Re}_x^{1/2} \qquad (\mathrm{Pr} \ll 1) \tag{69}$$

These results compare favorably with the scaling laws [Eqs. (37)-(40)]. They also compare favorably with more exact (and expensive) solutions.

$$\frac{\delta_T}{L} \sim \mathrm{Pe}_L^{-1/2} \sim \mathrm{Pr}^{-1/2}\,\mathrm{Re}_L^{-1/2}$$

$$\frac{\delta_T}{\delta} \sim \mathrm{Pr}^{-1/2} \gg 1$$

$$h \sim \frac{k}{L}\,\mathrm{Pr}^{1/2}\,\mathrm{Re}_L^{1/2} \qquad (\mathrm{Pr} \ll 1)$$

$$\mathrm{Nu} \sim \mathrm{Pr}^{1/2}\,\mathrm{Re}_L^{1/2}$$

Next session:

- *Laminar Boundary Layer Flow:*
  Similarity Solution