Sharif Quantum Information Group

Topological Quantum Computation

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New Advances in Condensed Matter Physics:
Quantum transport, topological effects and energy conversion in low-dimensional systems

Khiva, Uzbekistan, 2017

Objectives

- To understand the basic ideas of:

Topological Qubit

Topological Order

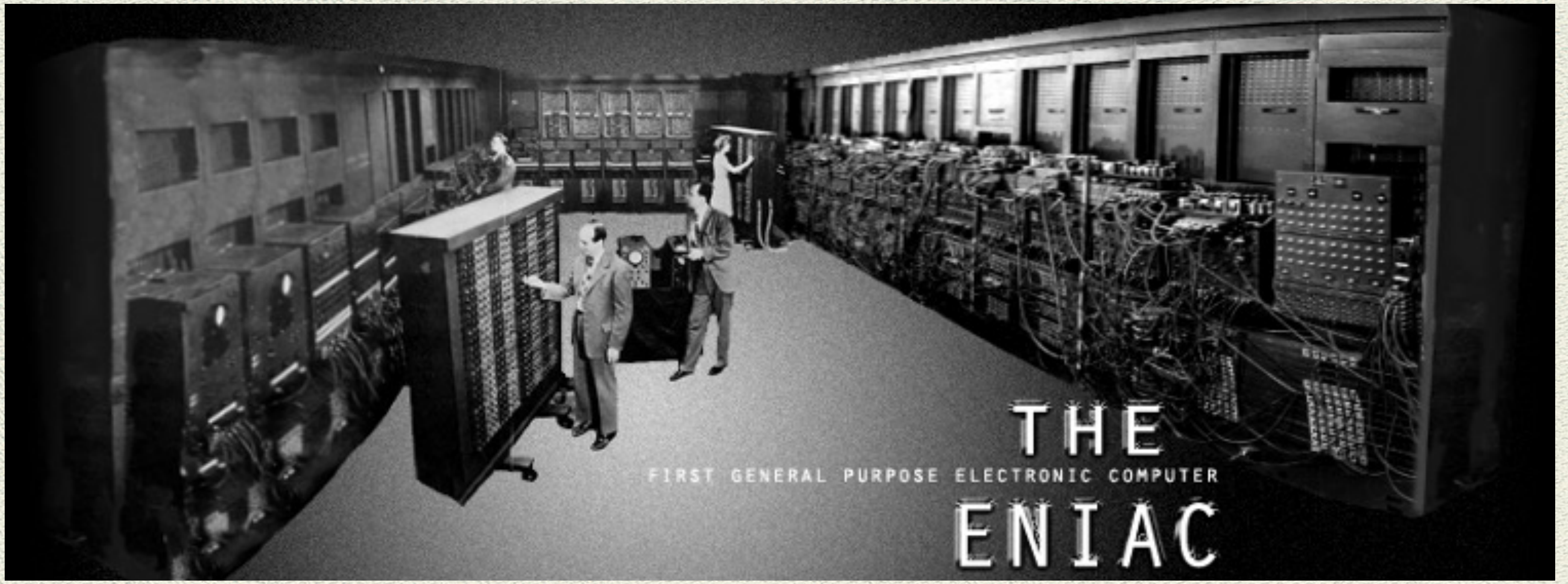
Kitaev Model

Topological Quantum Computation →

Classical Bits

Quantum Bits

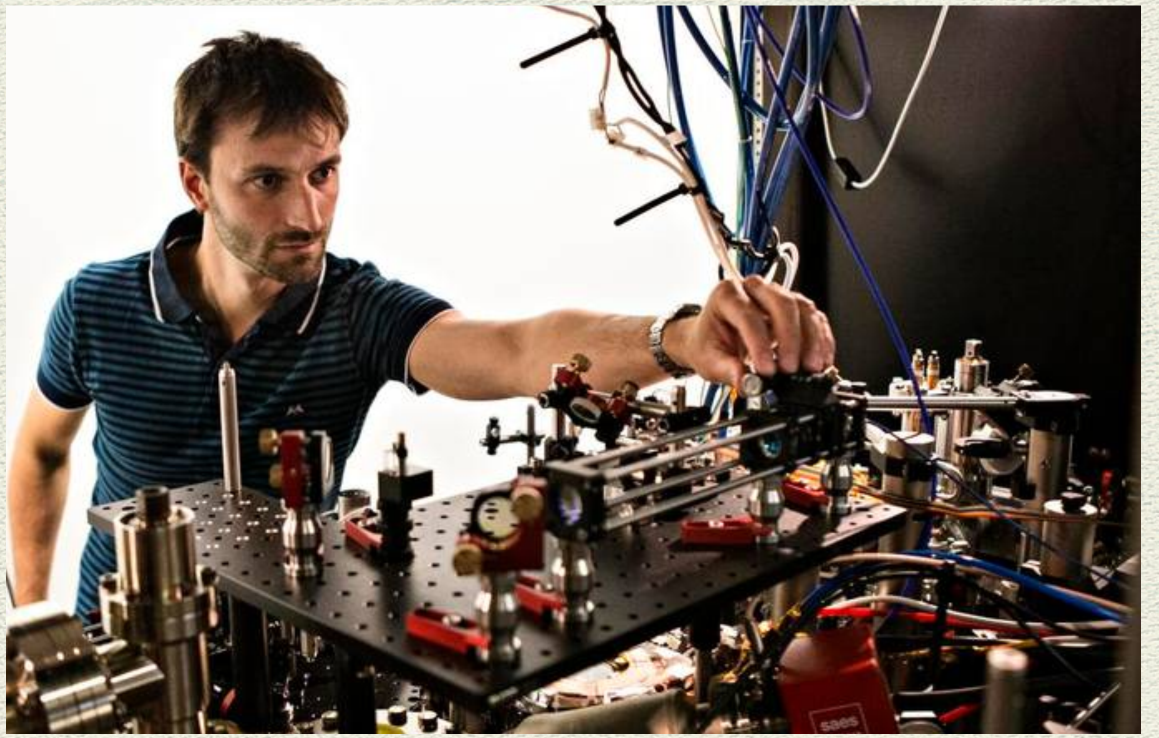
The early computers and computes of today



Intel Broadwell-EP Xeon

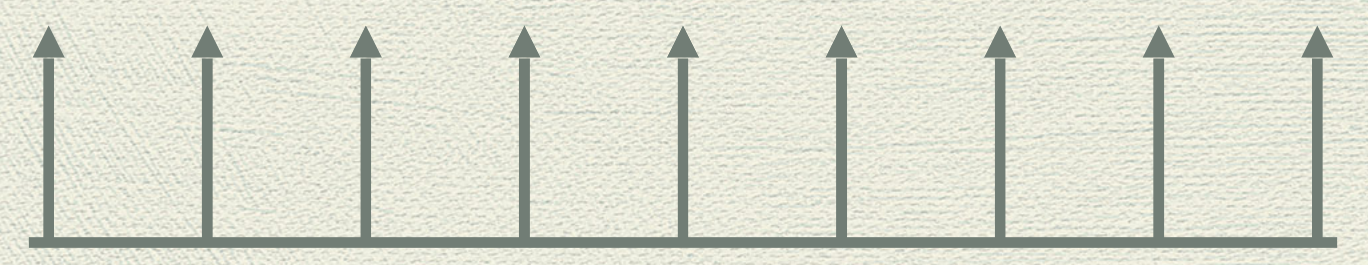
7.2 Billion Transistors

Computers of today and those of tomorrow

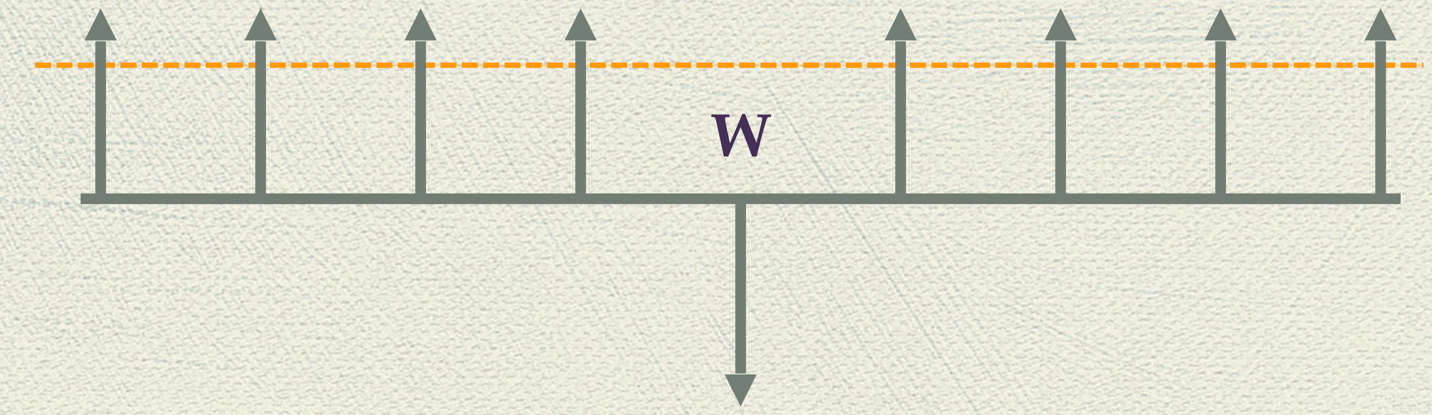


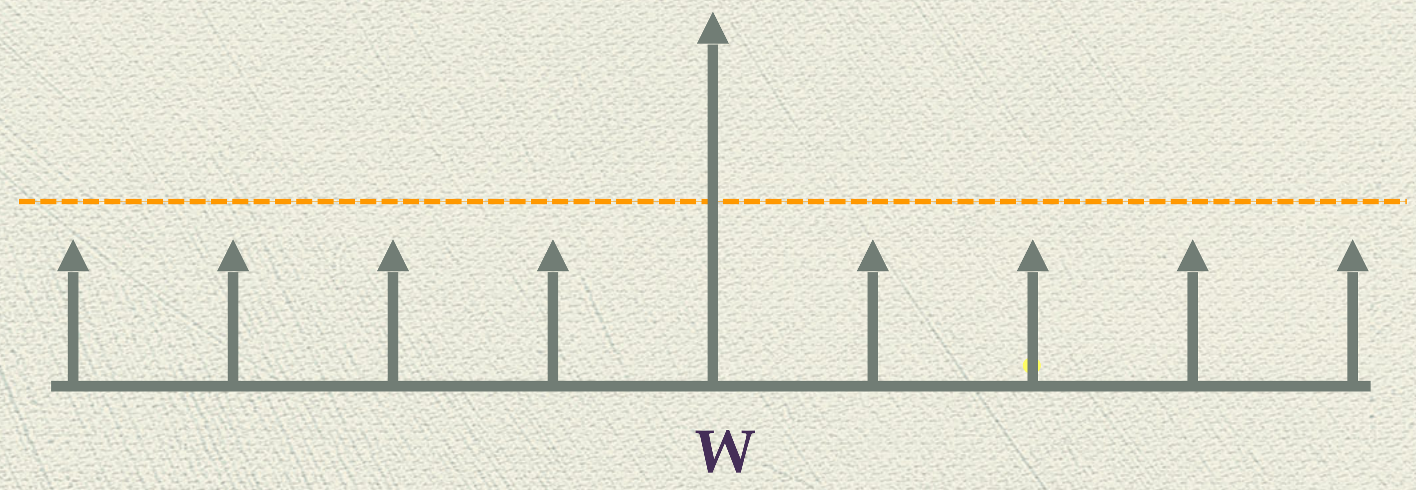
16 Giga Bytes	1 Kilo byte
3.8 GHz	1 KHz
1300 US dollars	10s of millions of dollars

$$\sum_{x=1}^{N} |x\rangle$$



$$\sum_{x=1}^{N} (-1)^{f(x)} |x\rangle$$



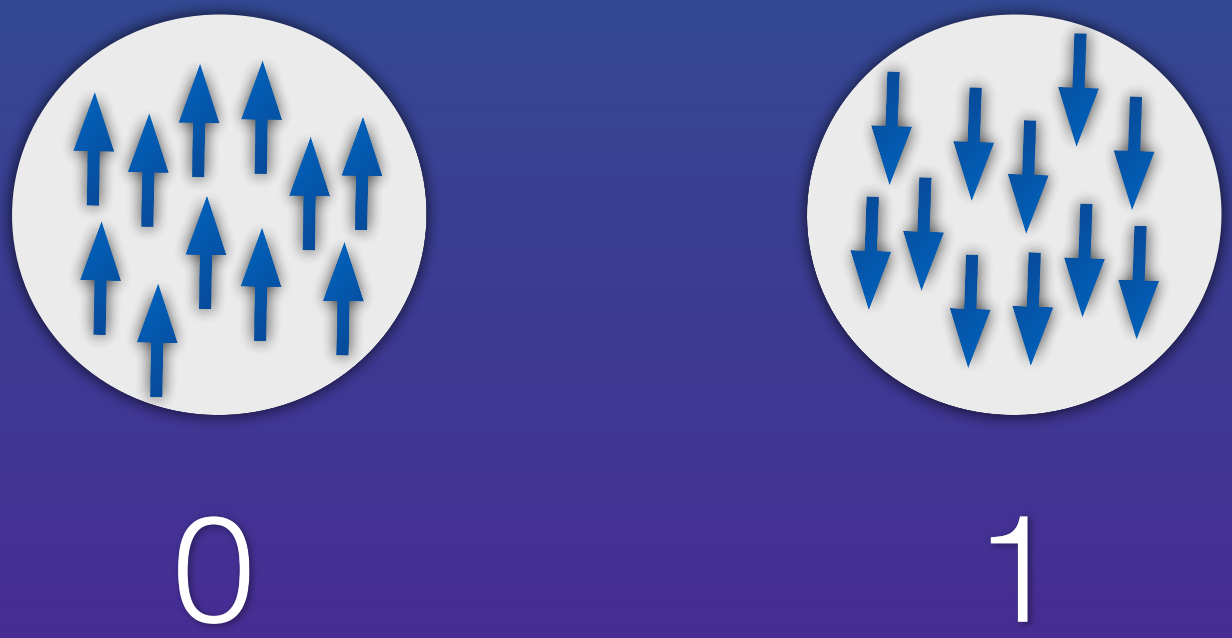


Classical Bits Have

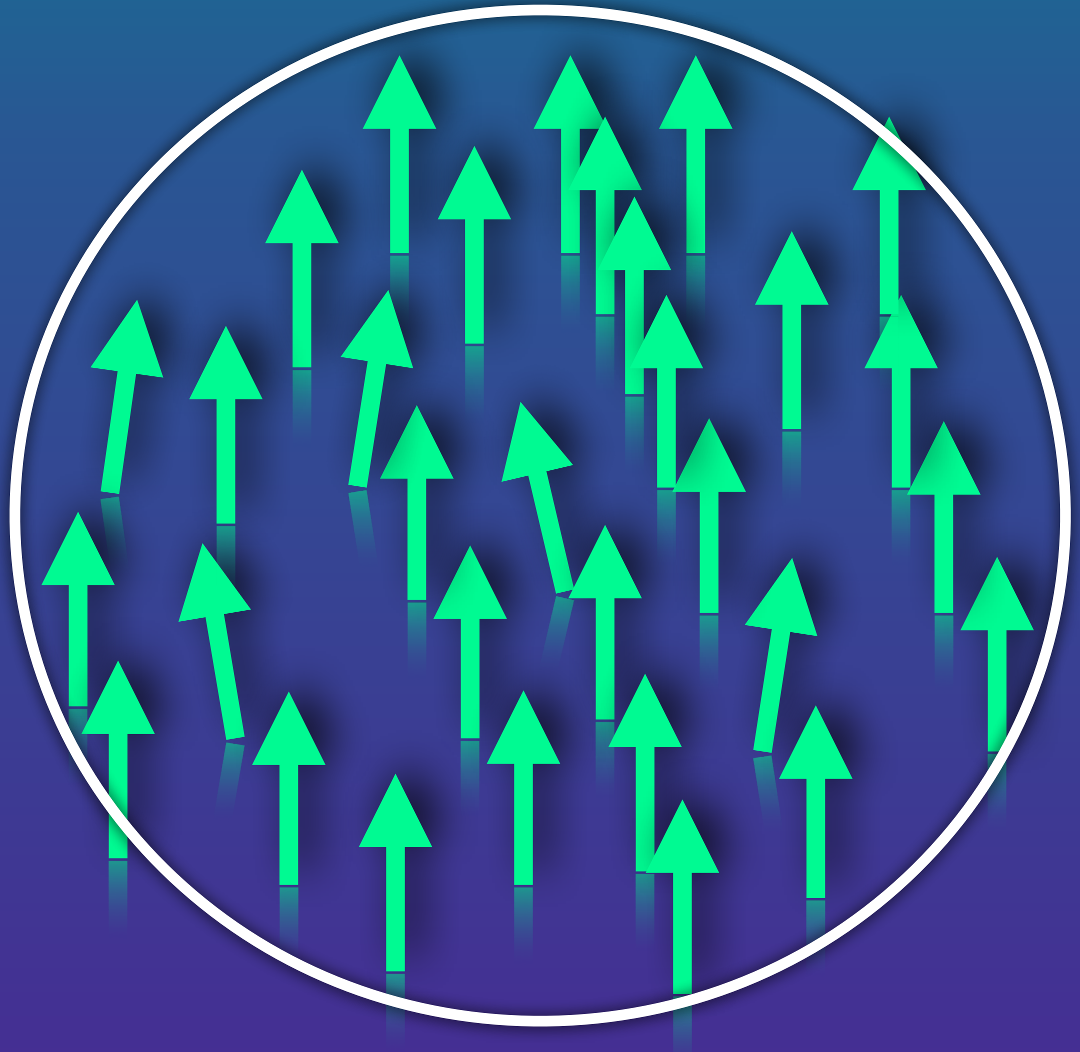
Four

Very Good Properties

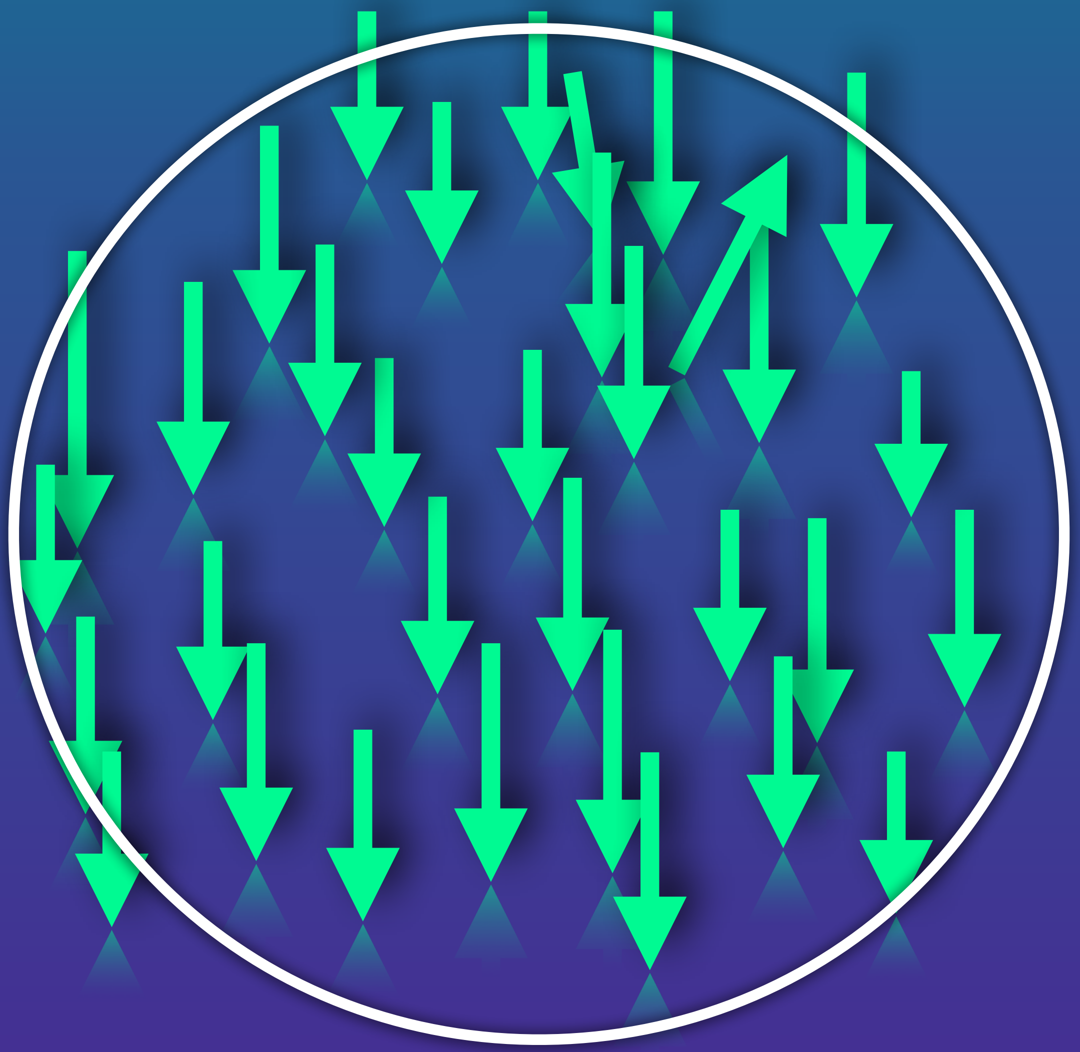
1- Bits are Macroscopic Objects



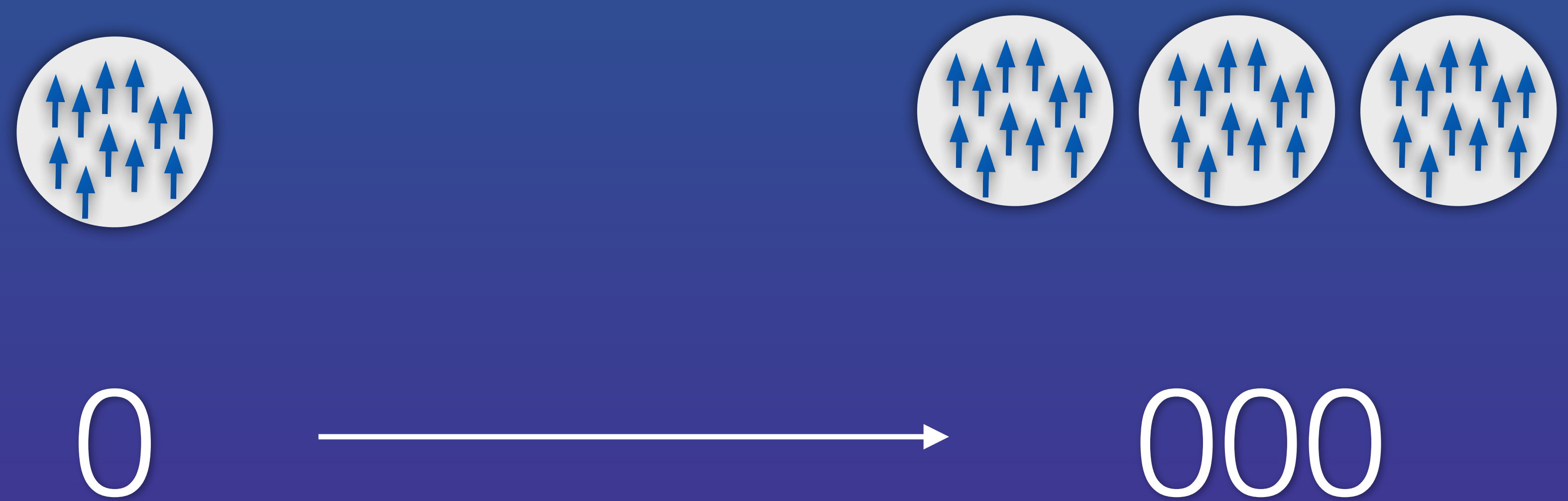
0



1



2- Bits can be cloned



3- Errors are discrete

0 ————————> 1

1 ————————> 0

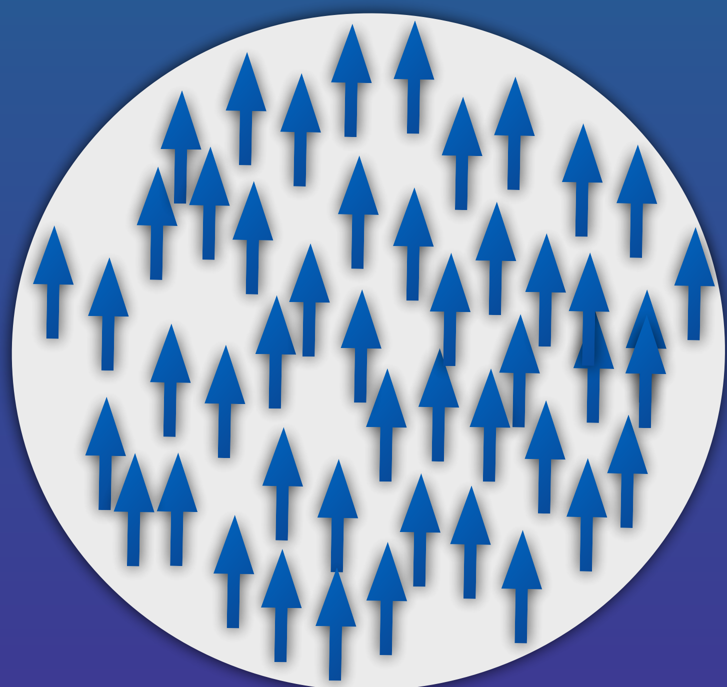
4- Bits can be observed

010 ⟶ 000

And corrected

Qubits have exactly the opposite properties

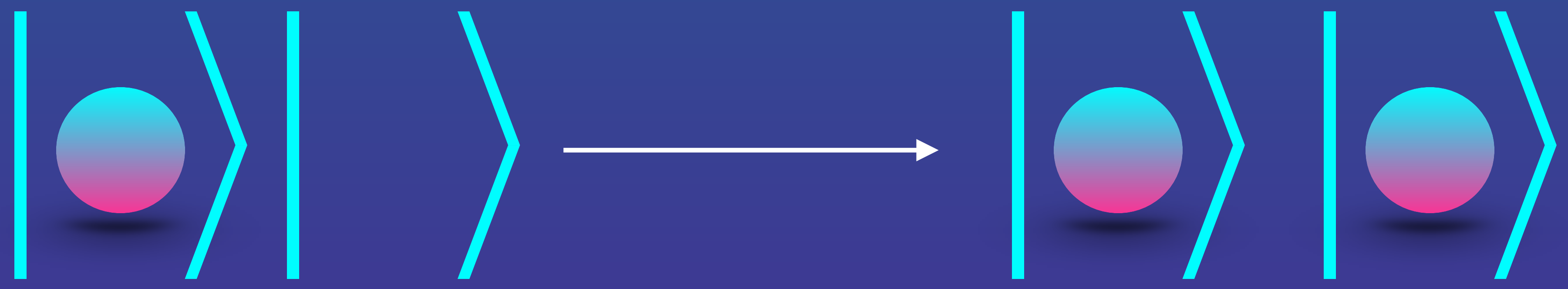
They are microscopic



Bit

Qubit

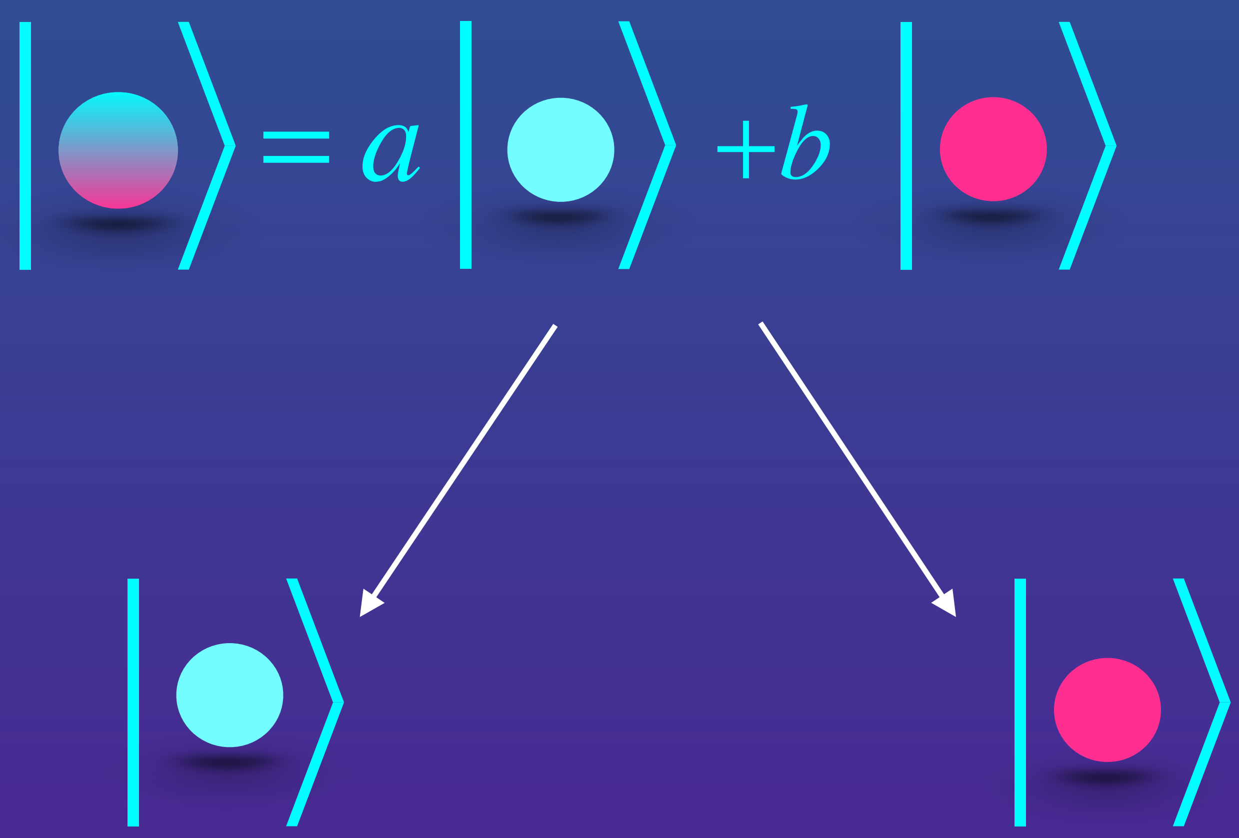
They cannot be cloned



$$|\psi\rangle \otimes |0\rangle \rightarrow |\psi\rangle \otimes |\psi\rangle$$

Quantum Errors are continuous

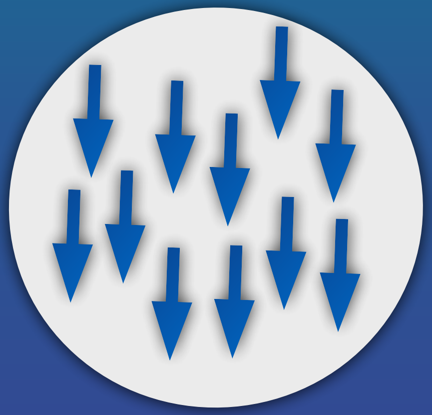
They cannot be observed



Topological Qubits

Merging the good features of both

$$|\overline{0}\rangle$$



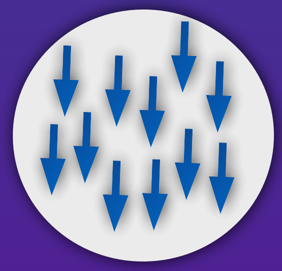
$$|\overline{1}\rangle$$

$$a|\overline{0}\rangle + b|\overline{1}\rangle$$

Ising Model

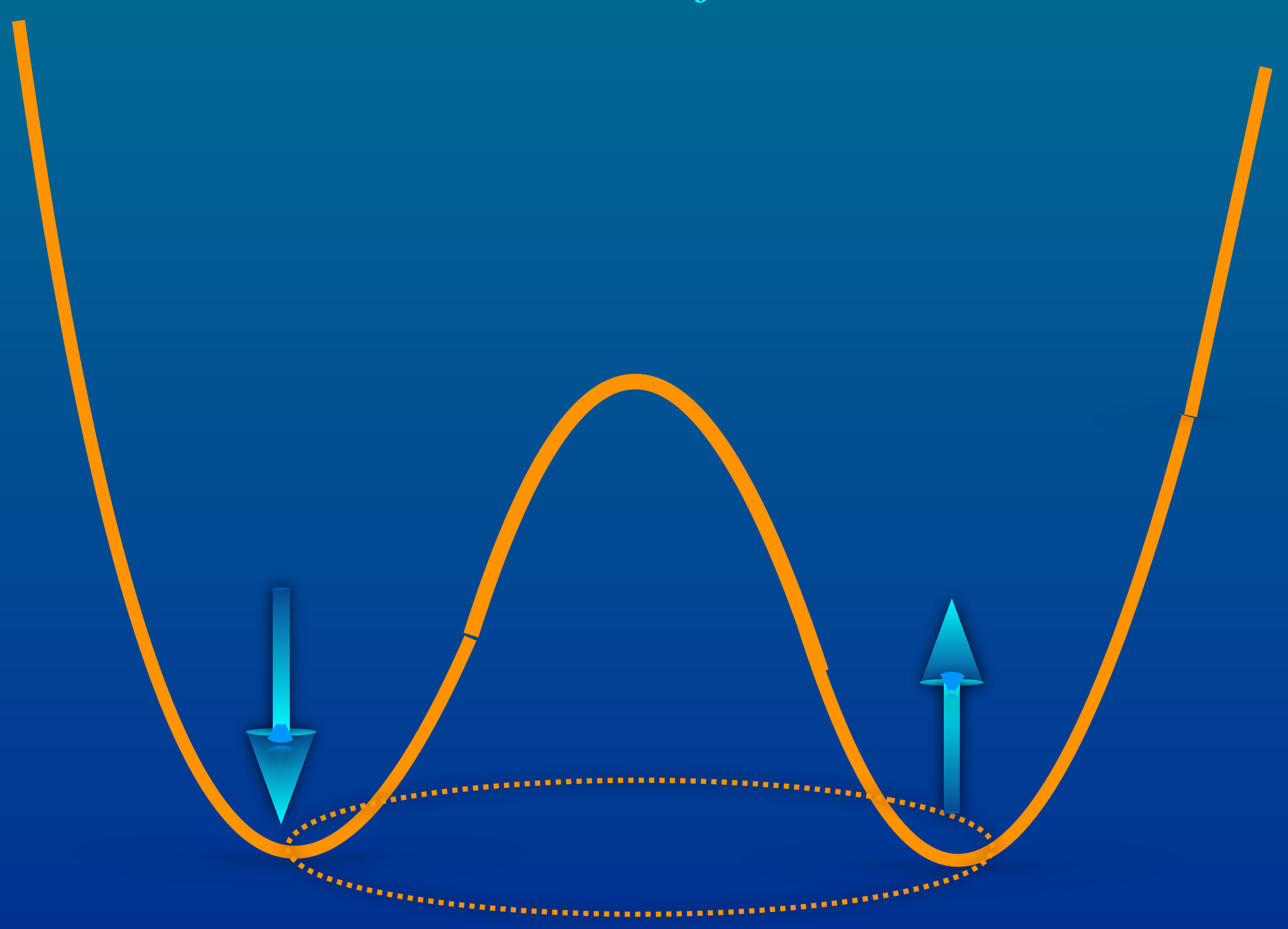
$$H = -\sum_{<i,j>} z_i z_j$$

$$\left|\bar{0}\right\rangle = \left|\uparrow\uparrow\uparrow \ldots\ldots \uparrow\uparrow\uparrow\right\rangle \qquad \left|\bar{1}\right\rangle = \left|\downarrow\downarrow\downarrow \ldots\ldots \downarrow\downarrow\downarrow\right\rangle$$



Symmetry Breaking

$$H = -\sum_{<i,j>} Z_i Z_j$$

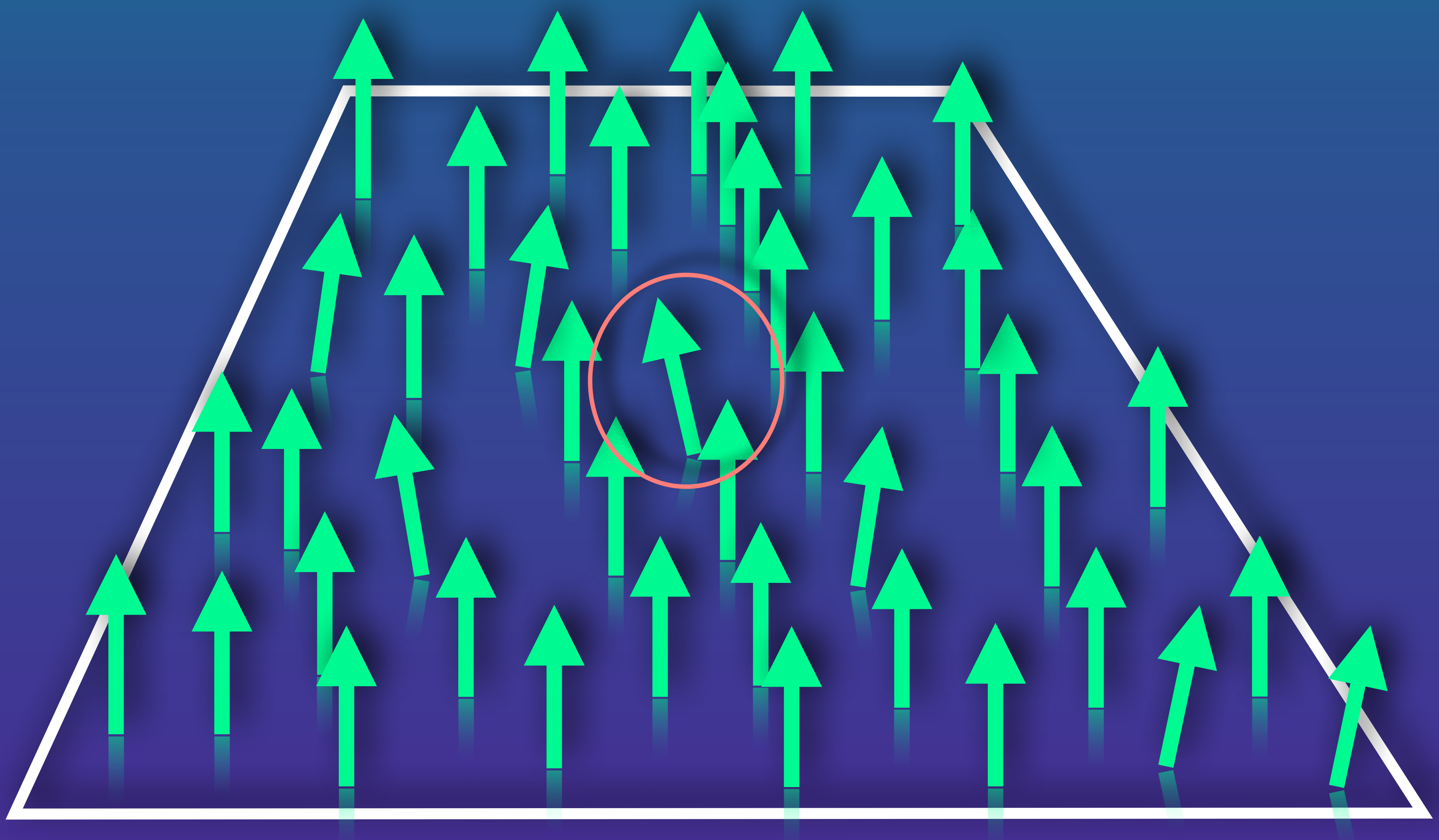


$$|00000000\rangle \qquad |1111111\rangle$$

Local Order

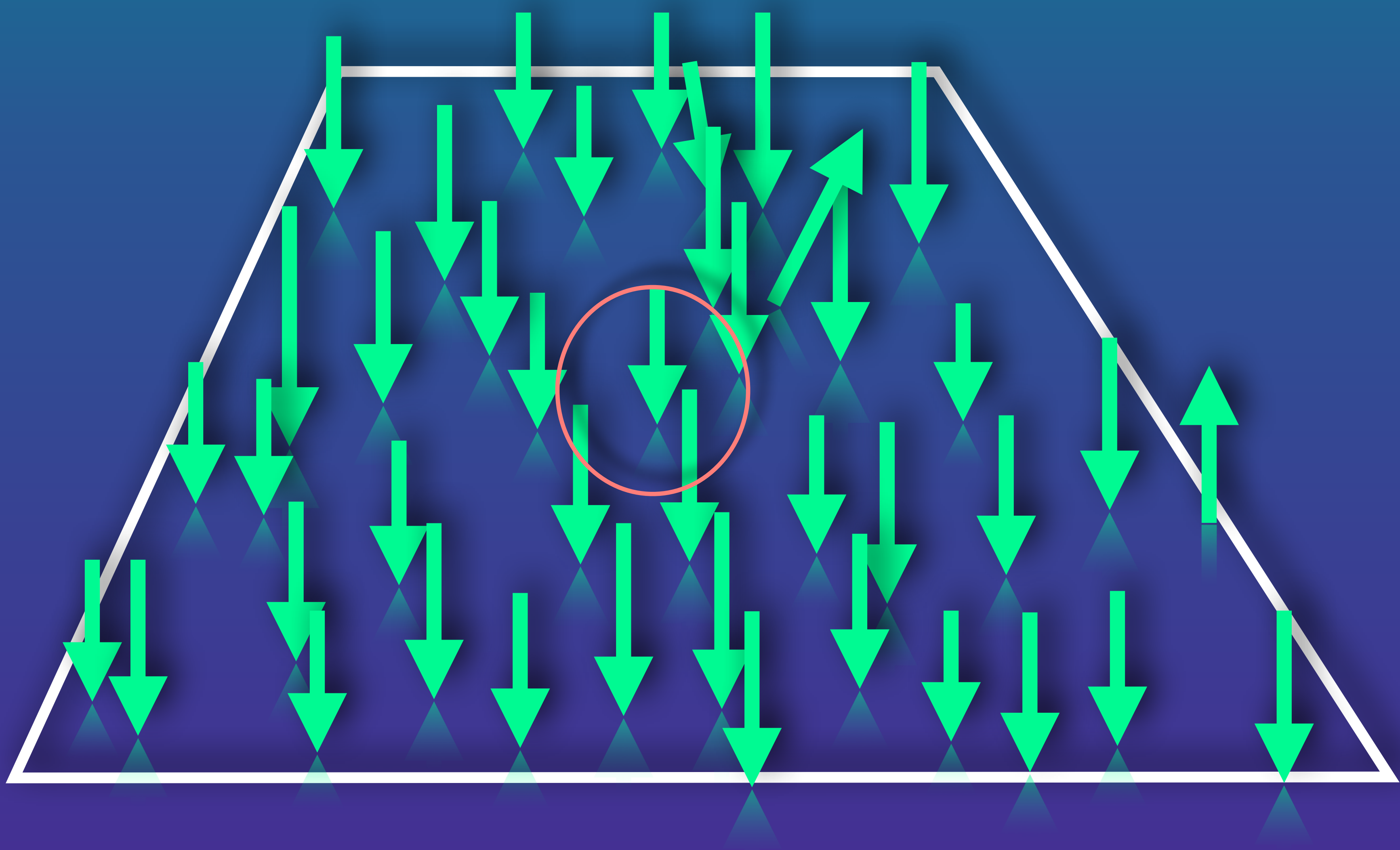
$$\langle \sigma_z \rangle = 1$$

$$|0\rangle$$



$$\langle \sigma_z \rangle = -1$$

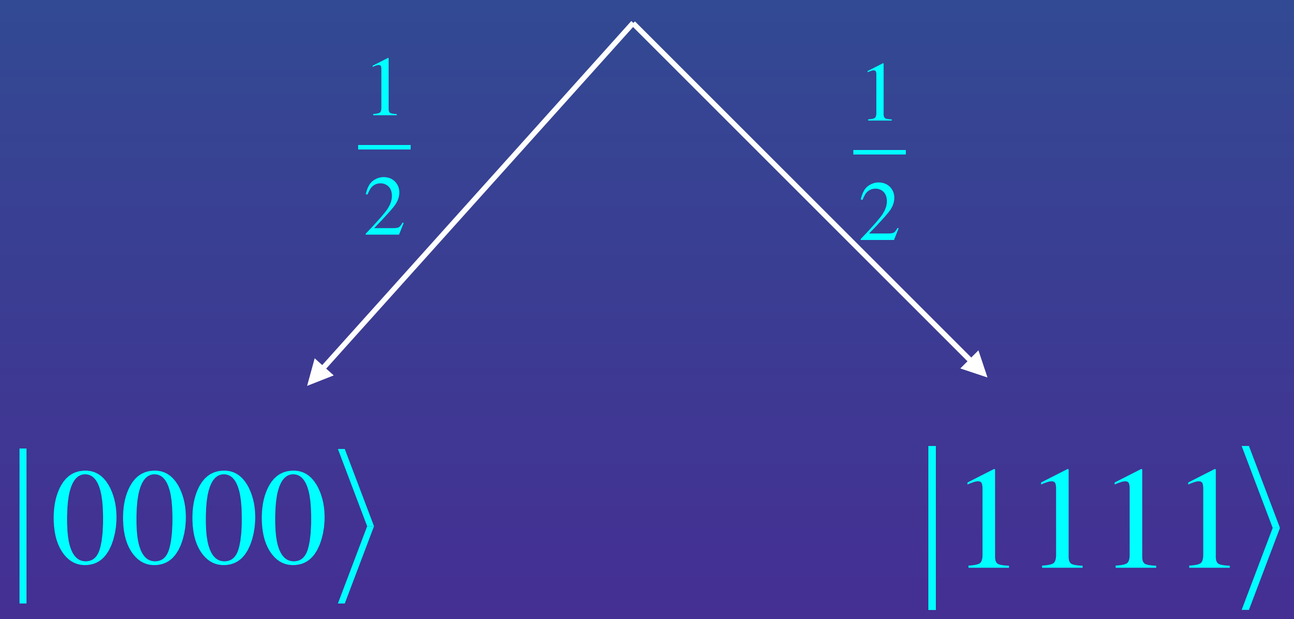
$$|1\rangle$$



But local order cannot produce a topological qubit!

Local order is extremely fragile

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|0000\rangle + |1111\rangle\right)$$



$$|W\rangle = \frac{1}{2}\left(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle\right)$$

$$\frac{1}{4} \qquad\qquad \frac{3}{4}$$

$$|W_1\rangle = |000\rangle \qquad\qquad |W_0\rangle = \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right)$$

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|00\ldots\ldots00\rangle + |11\ldots\ldots11\rangle\right)$$

$$|00\ldots\ldots00\rangle \qquad\qquad |11\ldots\ldots11\rangle$$

$$|W\rangle = \frac{1}{\sqrt{N}}\left(|100....000\rangle + |010....000\rangle + |001....000\rangle + ... + |00....001\rangle\right)$$

$$\Big\downarrow \; \frac{N-1}{N}$$

$$|W_0\rangle = \frac{1}{\sqrt{N-1}}\left(|100....00\rangle + |010....00\rangle + |001....00\rangle + ... + |00....01\rangle\right)$$

What we want?

1-Degenerate ground state

2-Existence of Gap

3-Not locally distinguishable

4-Robust to perturbations

A system with degenerate ground state

$$|\psi_0\rangle \qquad\qquad |\psi_1\rangle$$

which cannot be distinguished,
by any local observable!

$$\langle\psi_0|K|\psi_0\rangle = \langle\psi_1|K|\psi_1\rangle$$

The Stabilizer Formalism

$$|GHZ\rangle = |000\rangle + |111\rangle$$

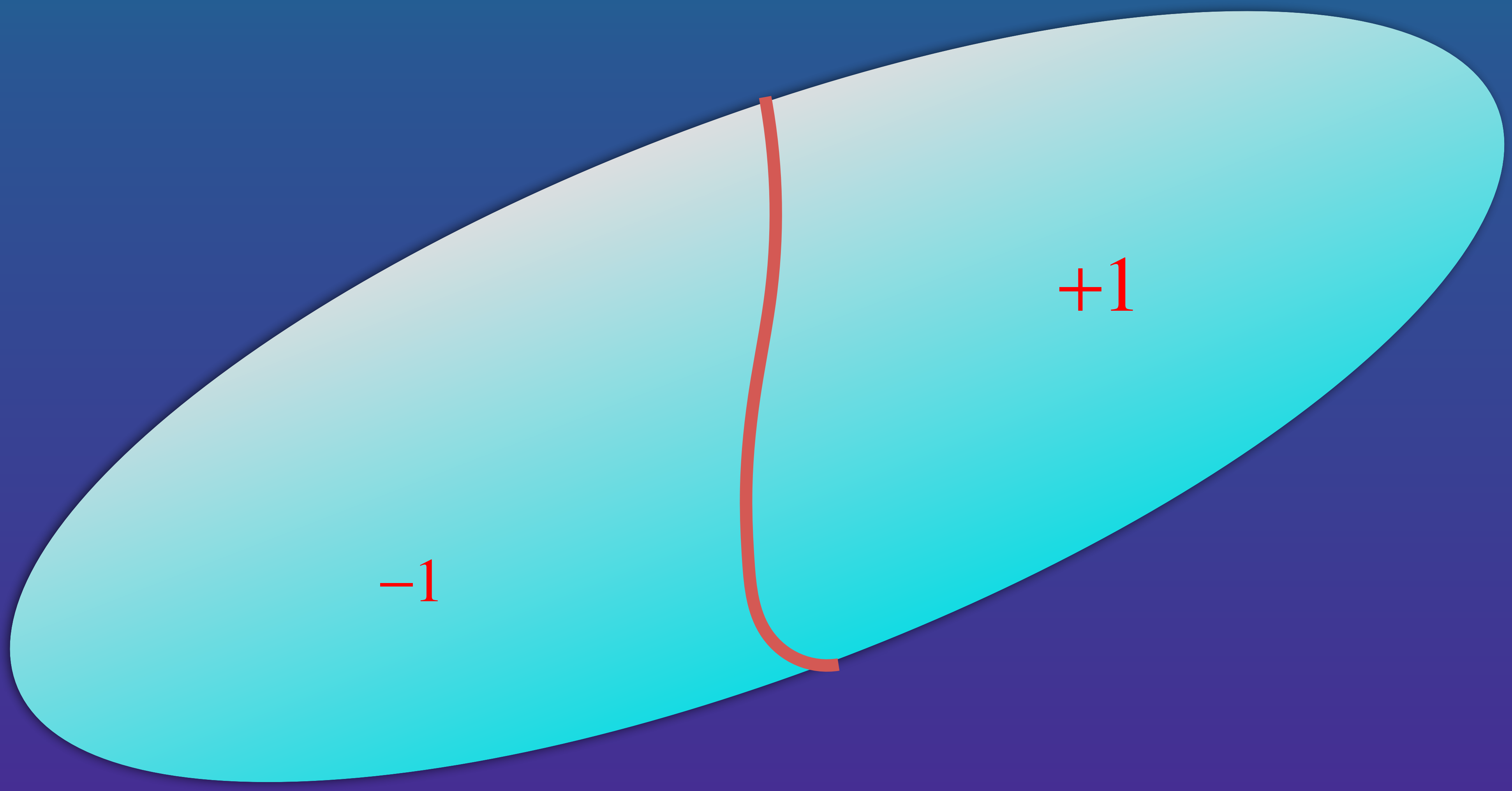
$$X_1 X_2 X_3 |GHZ\rangle = |GHZ\rangle$$

$$Z_1 Z_2 |GHZ\rangle = |GHZ\rangle$$

$$S = \{I, Z_1 Z_2, Z_1 Z_3, X_1 X_2 X_3\}$$

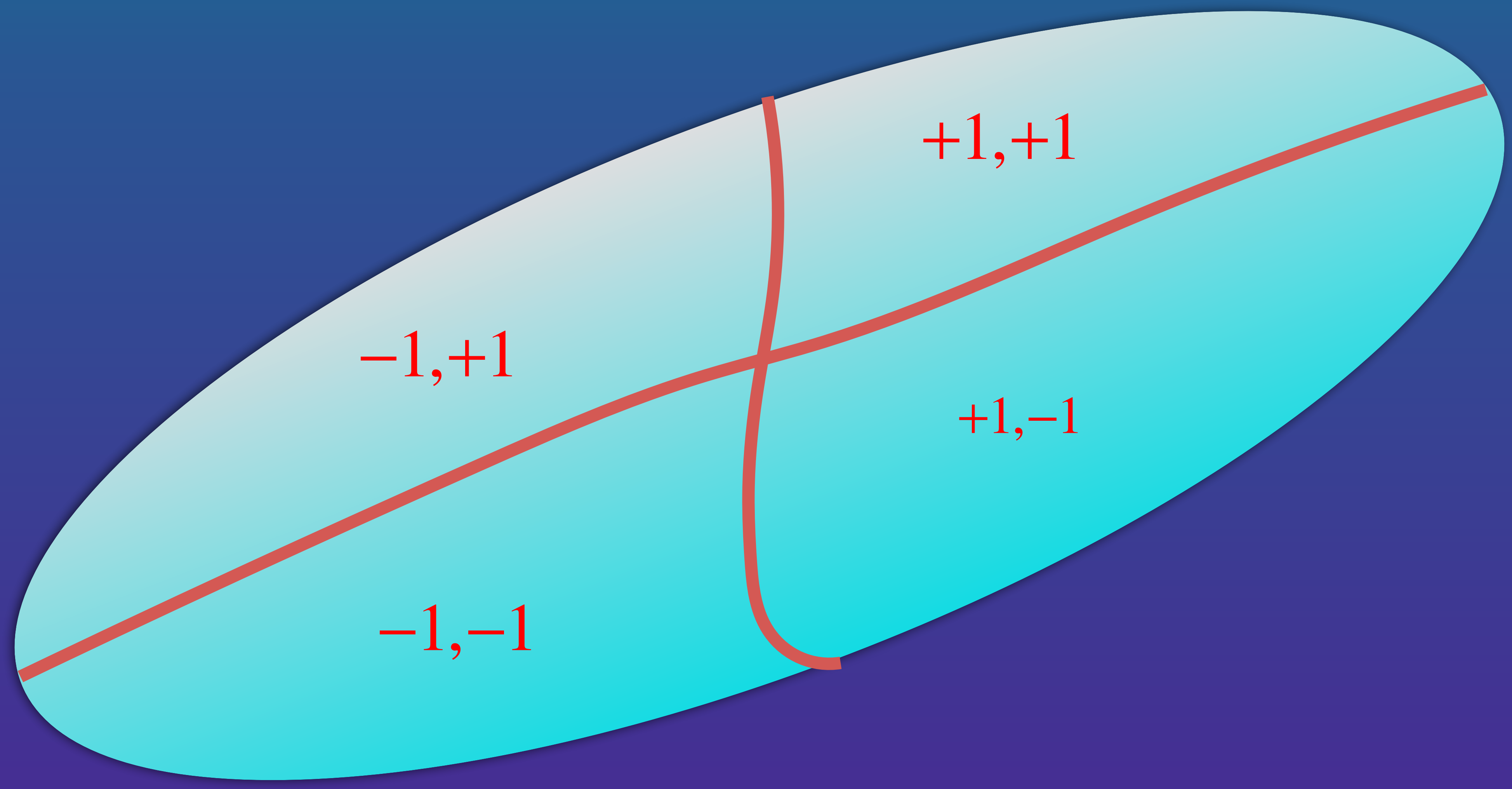
The Stabilizer Formalism

S_1



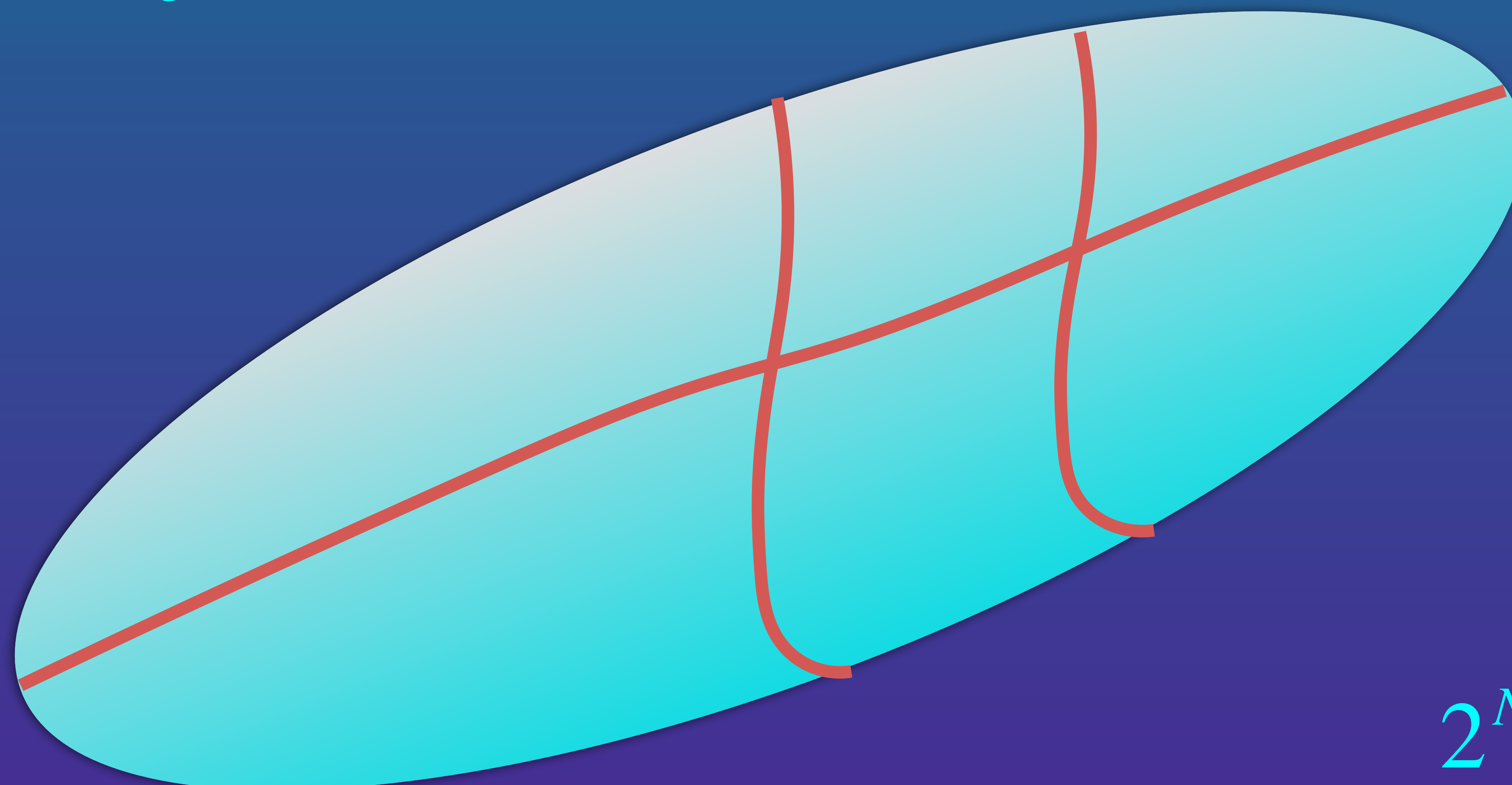
The Stabilizer Formalism

$S_1 \quad S_2$



The Stabilizer Formalism

$S_1 \quad S_2 \quad S_3$

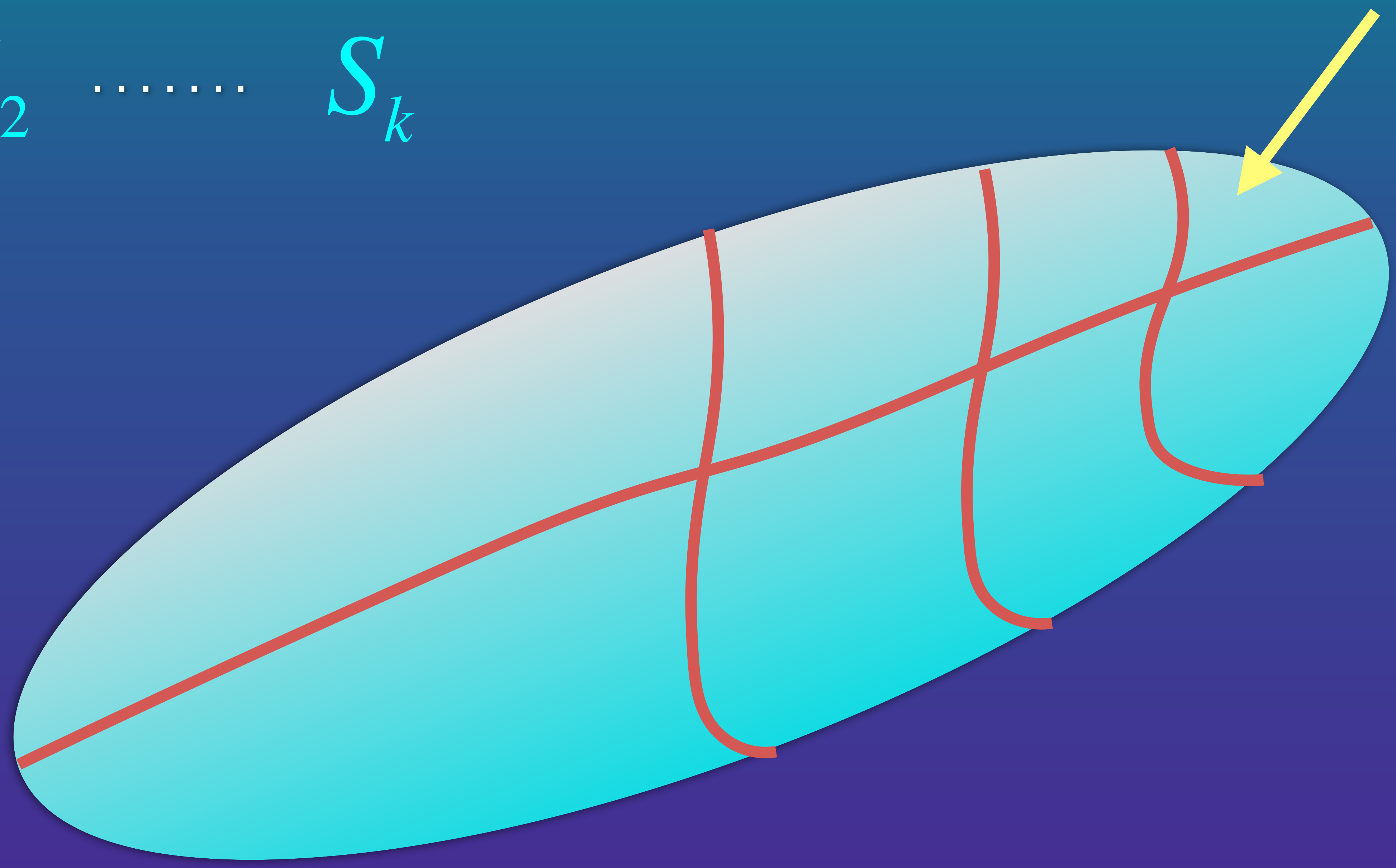


$$\frac{2^N}{2*2*2}$$

The Stabilizer Formalism

$S_1 \; S_2 \; \cdots\cdots \; S_k$

$$\frac{2^N}{2^k}$$



The Hamiltonian

$$H = -S_1 - S_2 - \dots\dots - S_k$$

All the local operators

$$H\left|\psi\right\rangle = -k\left|\psi\right\rangle$$

A ground state

The order of degeneracy

$$\frac{2^N}{2^k} = 2^{N-k}$$

The Kitaev Model

Notations

$$x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \left\{ \begin{array}{l} |+\rangle \\ |-\rangle \end{array} \right.$$

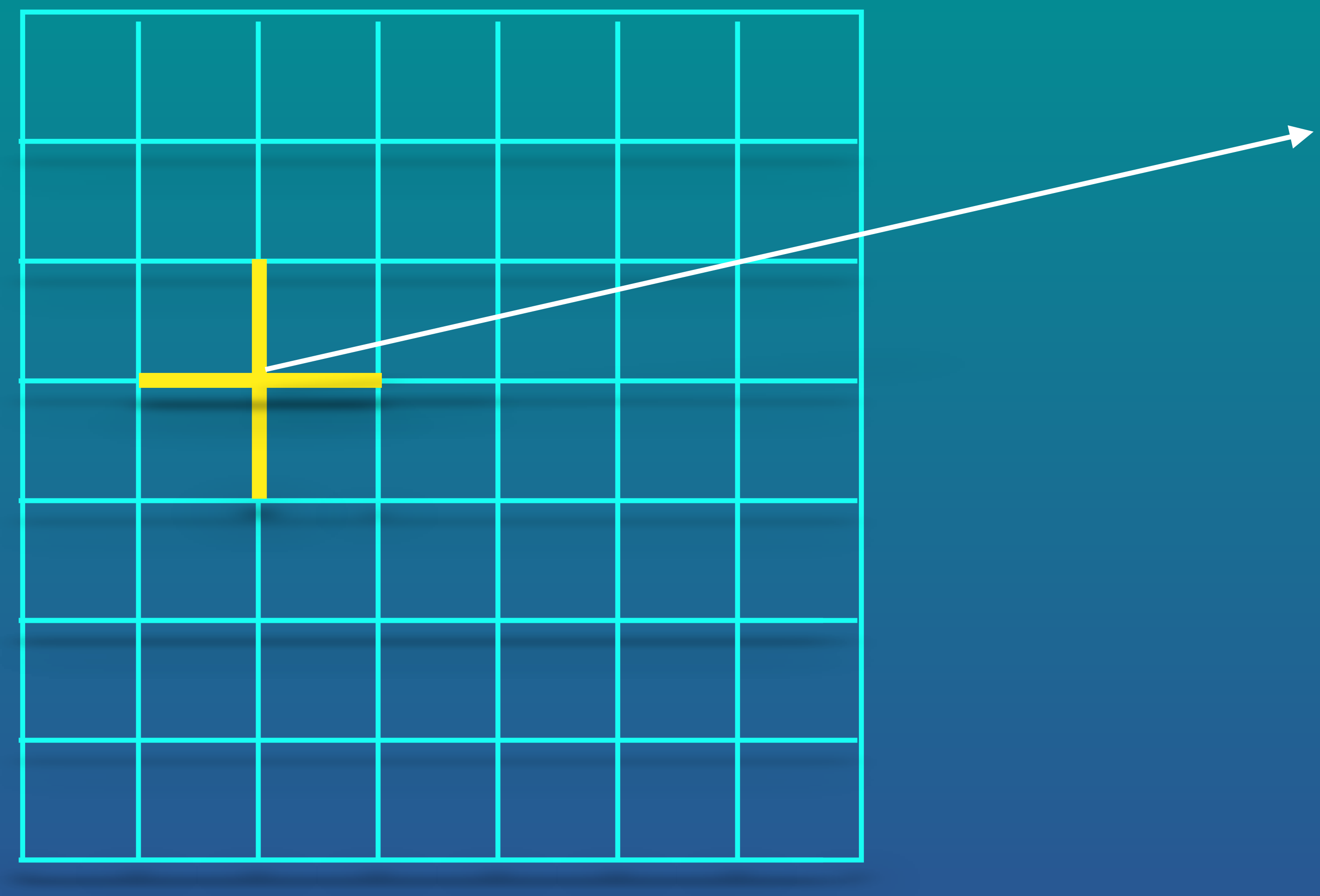
$$z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \left\{ \begin{array}{l} |0\rangle \\ |1\rangle \end{array} \right.$$

$$z|+\rangle = |-\rangle$$

$$x|0\rangle = |1\rangle$$

$$z|-\rangle = |+\rangle$$

$$x|1\rangle = |0\rangle$$



$$A_s = x_1 x_2 x_3 x_4$$

$$A_s^2 = I$$

$$\prod_s A_s = I$$

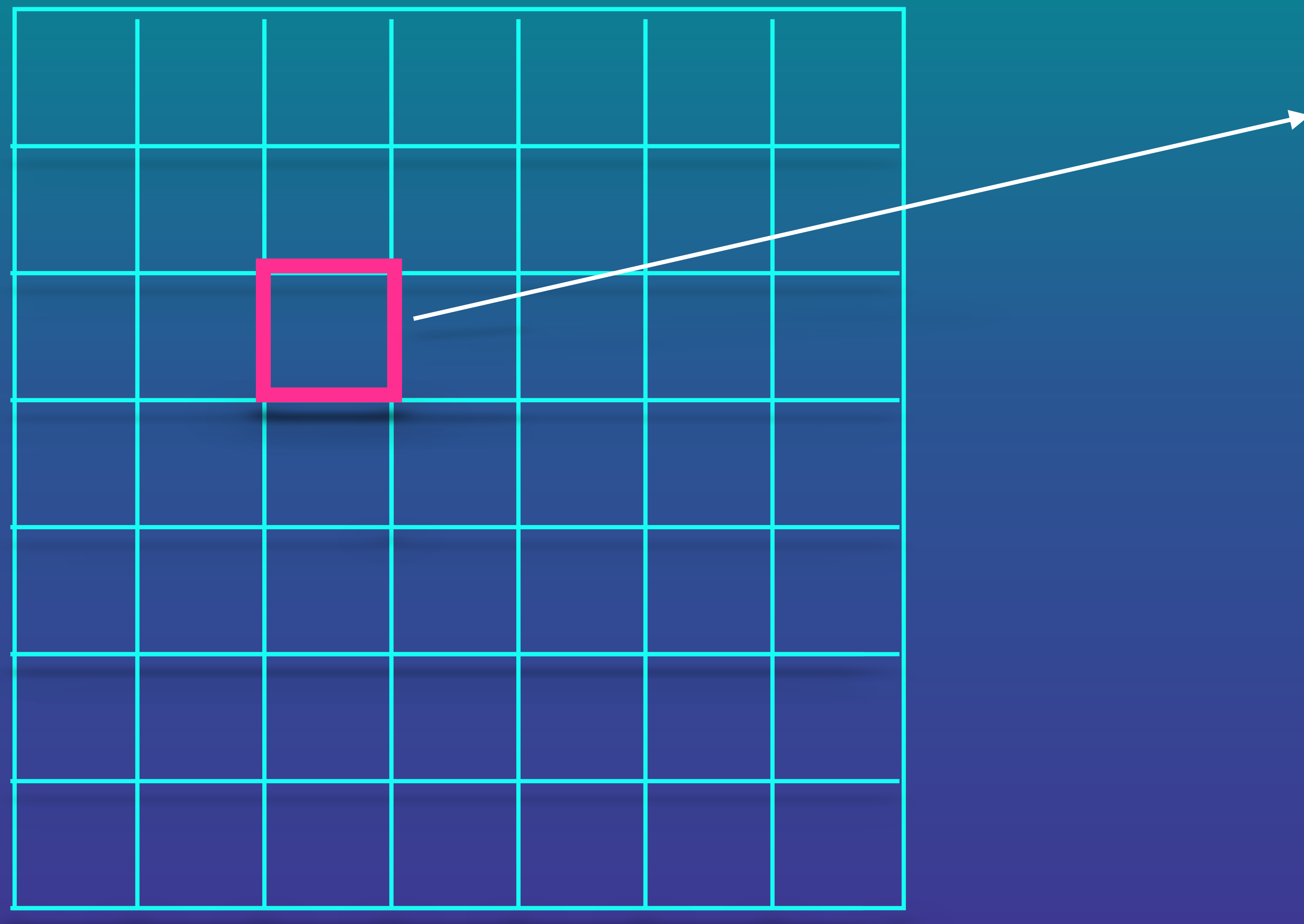
Number of vertices=N

Number of links=2N

Number of independent A's = N-1

$$Dimension\ \ \text{of}\ \ Hilbert\ Space = 2^{2N}$$

$$Degeneracy = \frac{2^{2N}}{2^{N-1}} = 2^{N+1}$$



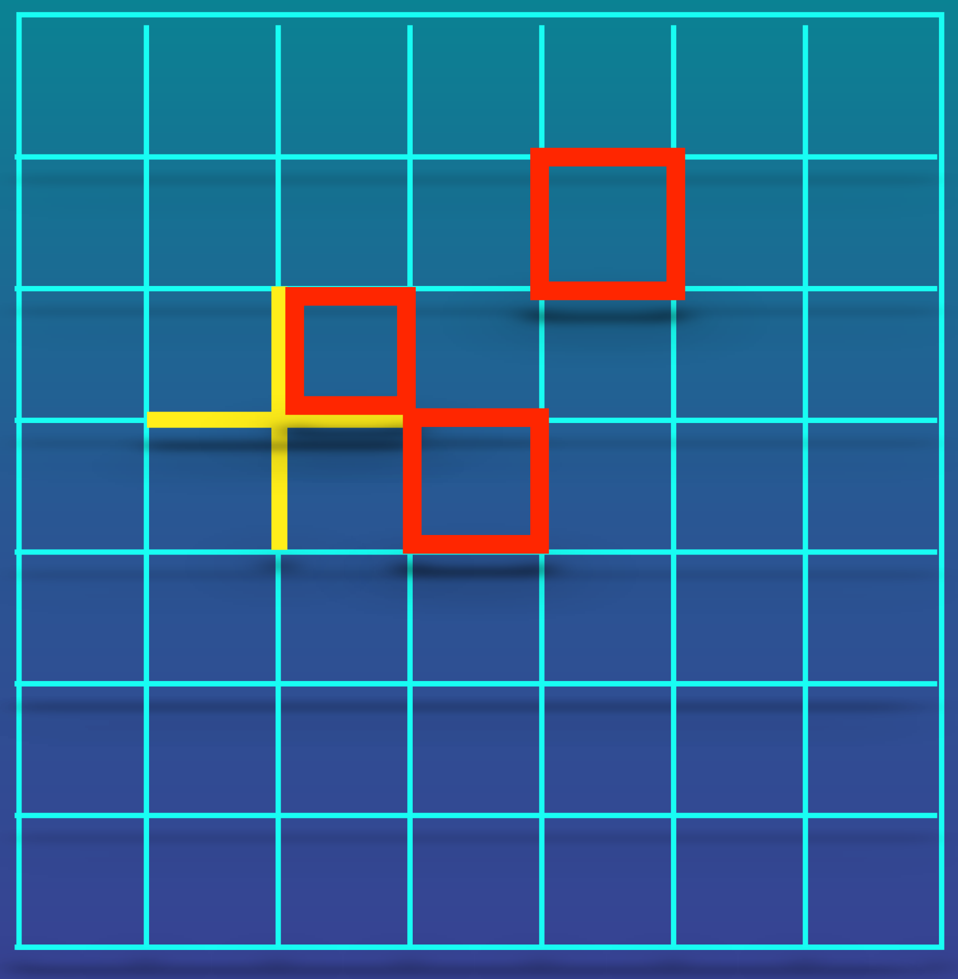
$$B_p = z_1 z_2 z_3 z_4$$

$$B_p^2 = I$$

$$\prod_p B_p = I$$

Number of faces = N

Number of Independent B’s = N-1



$$\left[A_s, B_p\right] = 0$$

$$H = -\sum_s A_s - \sum_p B_p$$

$$Degeneracy = \frac{2^{2N}}{2^{2N-2}} = 4$$

The ground state

$$H = -\sum_s A_s - \sum_p B_p$$

$$A_s|\phi\rangle = |\phi\rangle \qquad B_p|\phi\rangle = |\phi\rangle$$

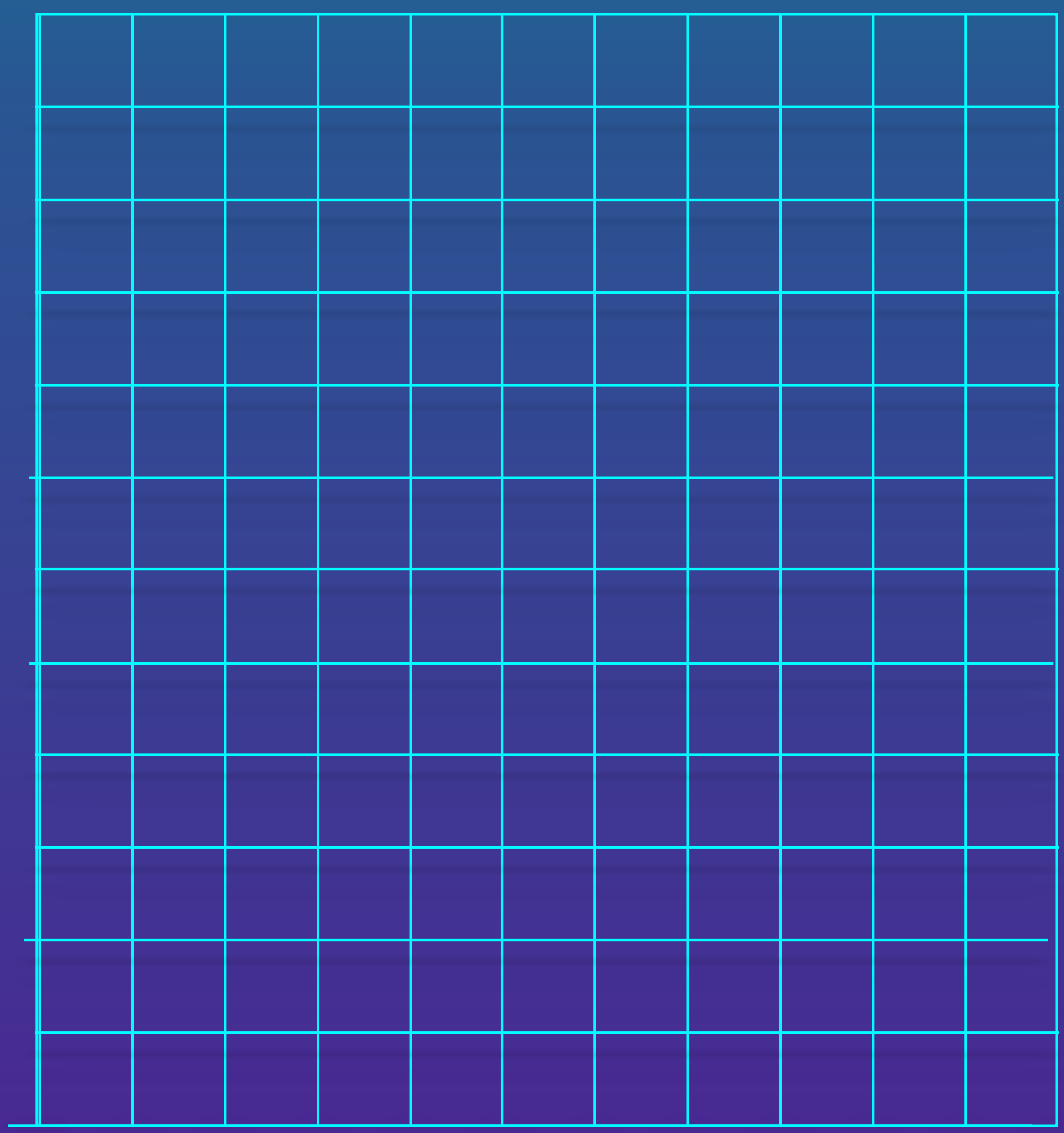
The ground state?

$$|\Omega\rangle = |+\rangle^{\otimes N}$$

$$A_s|\Omega\rangle = |\Omega\rangle$$

$$z|+\rangle = |-\rangle$$

$$B_p|\Omega\rangle \neq |\Omega\rangle$$



$$B_p(1+B_p)=B_p+1$$

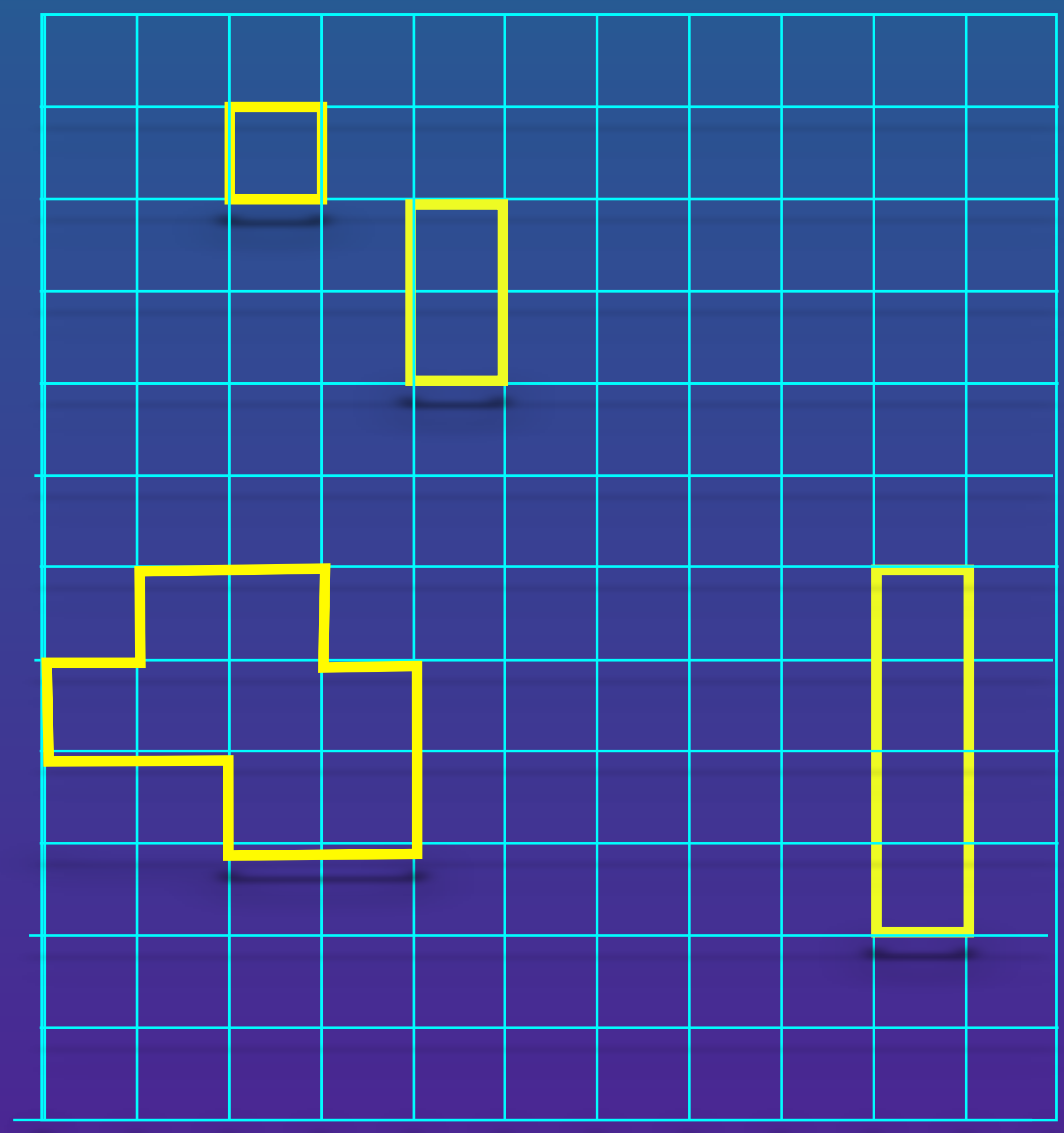
$$|\phi\rangle=(1+B_p)|\Omega\rangle$$

$$A_s|\phi\rangle=|\phi\rangle$$

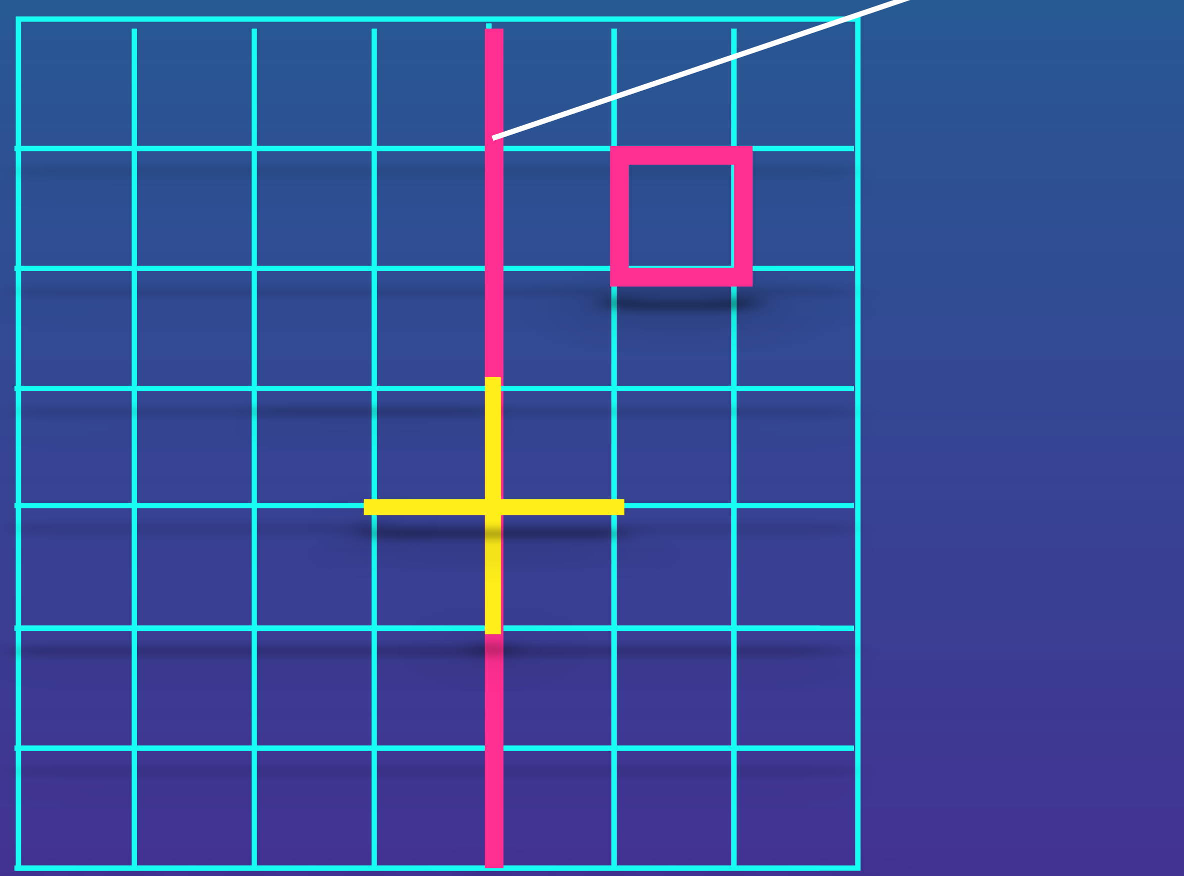
$$B_p|\phi\rangle=|\phi\rangle$$

$$|\varphi_0\rangle=\prod_p(1+B_p)|\Omega\rangle$$

What the ground state looks like?

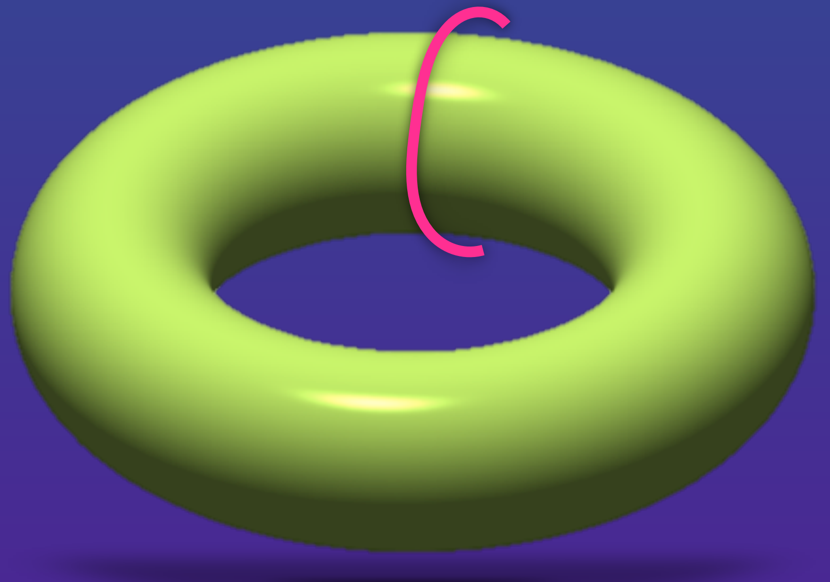


Where is the degeneracy?

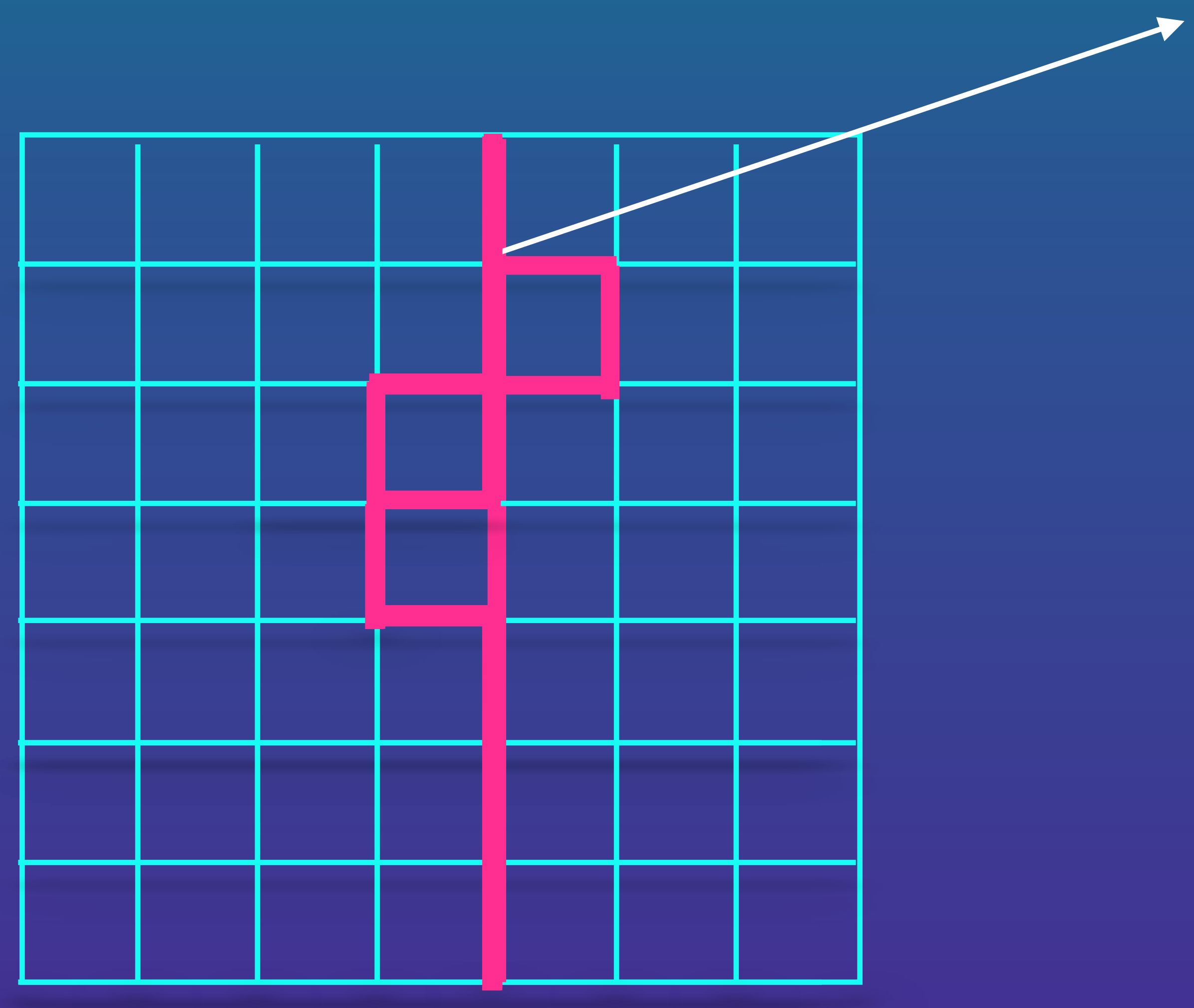


$$Z_1 = \prod_{i \in C_1} z_i$$

$$[Z_1 \, , \, \mathrm{H}] = 0$$

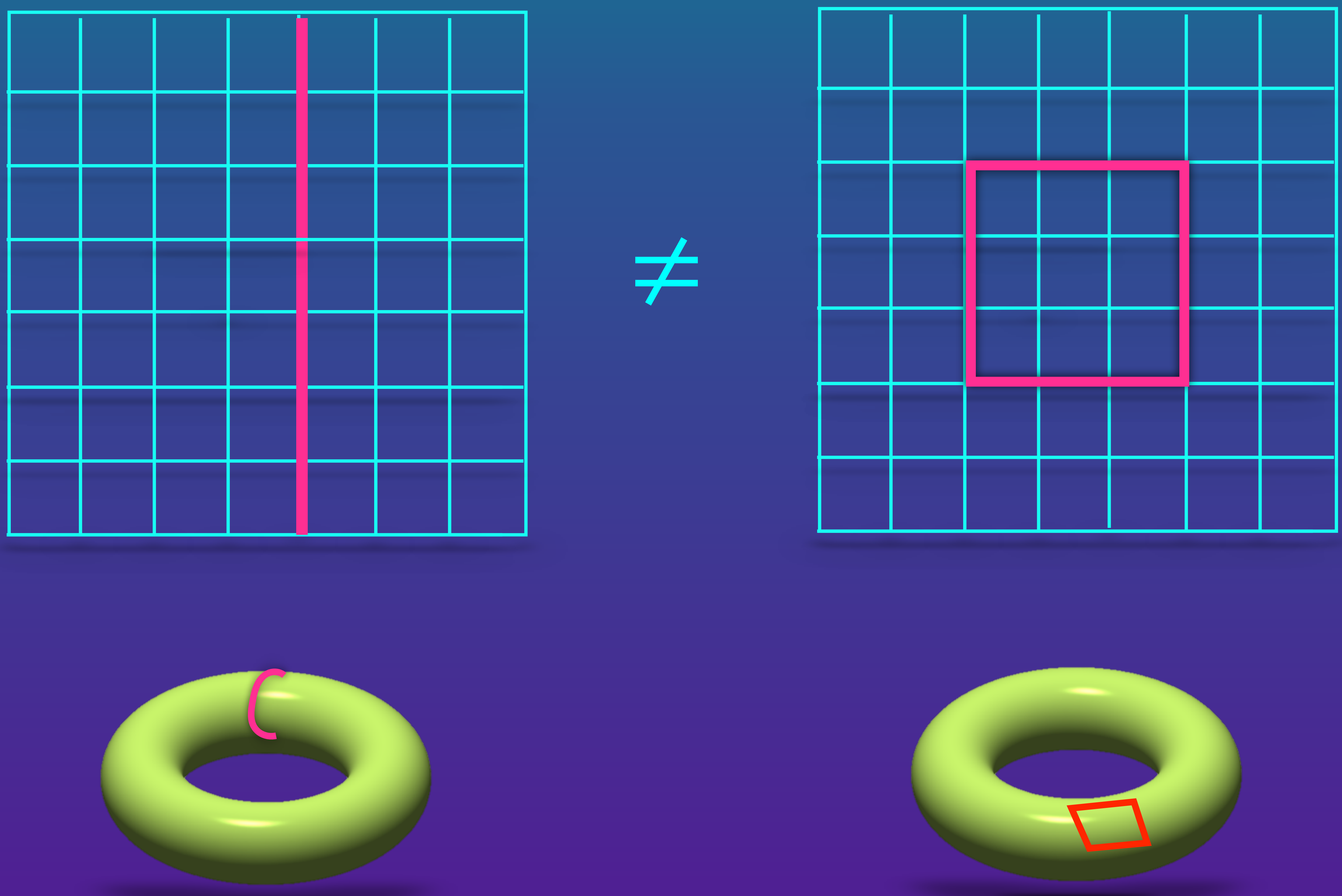


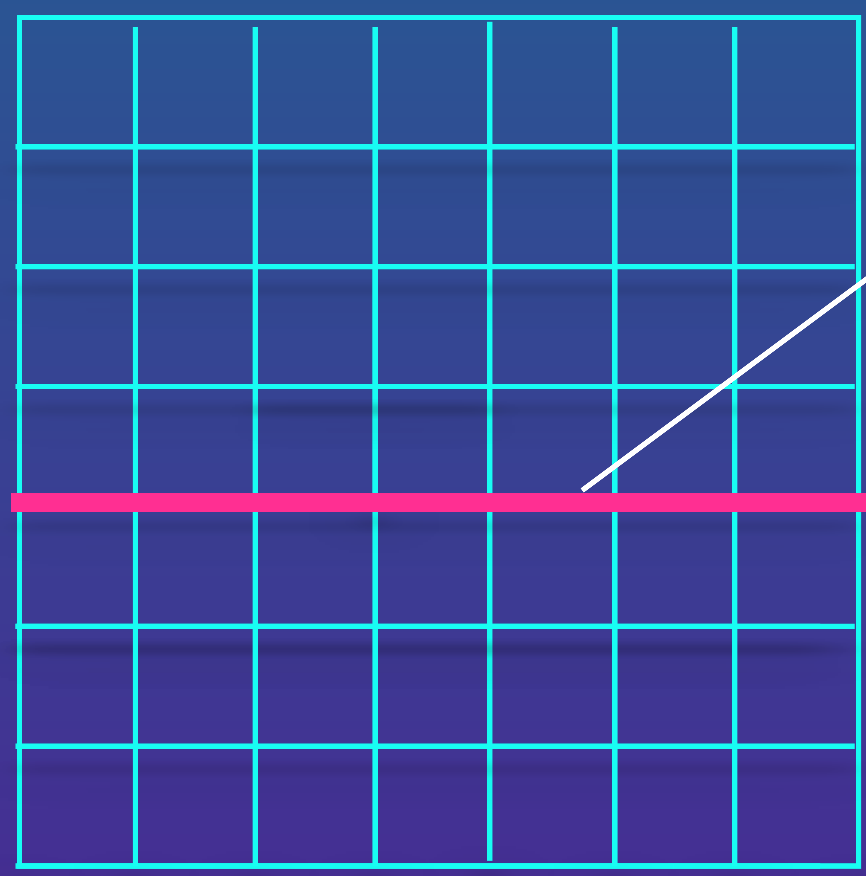
The line can be deformed



$$Z_1 = \prod_{i \in C_1} z_i$$

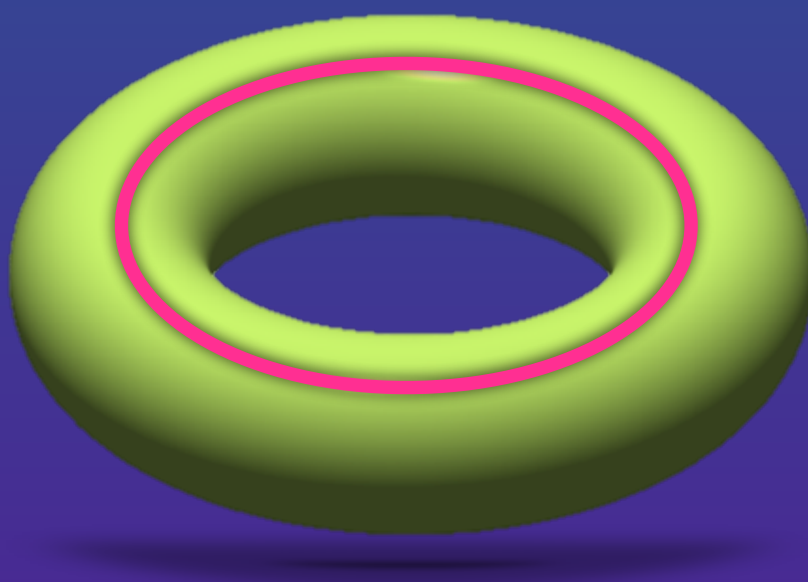
Where is the degeneracy?



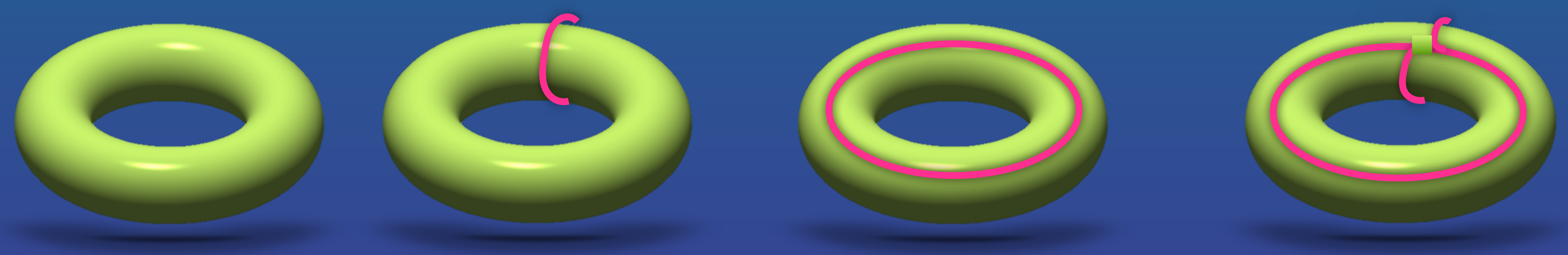


$$Z_2 = \prod_{i \in C_2} Z_i$$

$$[Z_2 \, , \, \mathrm{H}] = 0$$



Four ground states



$$|\phi_{00}\rangle \qquad |\phi_{01}\rangle = Z_1|\phi_{00}\rangle \qquad |\phi_{10}\rangle = Z_2|\phi_{00}\rangle \qquad |\phi_{11}\rangle = Z_1 Z_2|\phi_{00}\rangle$$

We achieved goal no. 1

Degenerate Ground States.

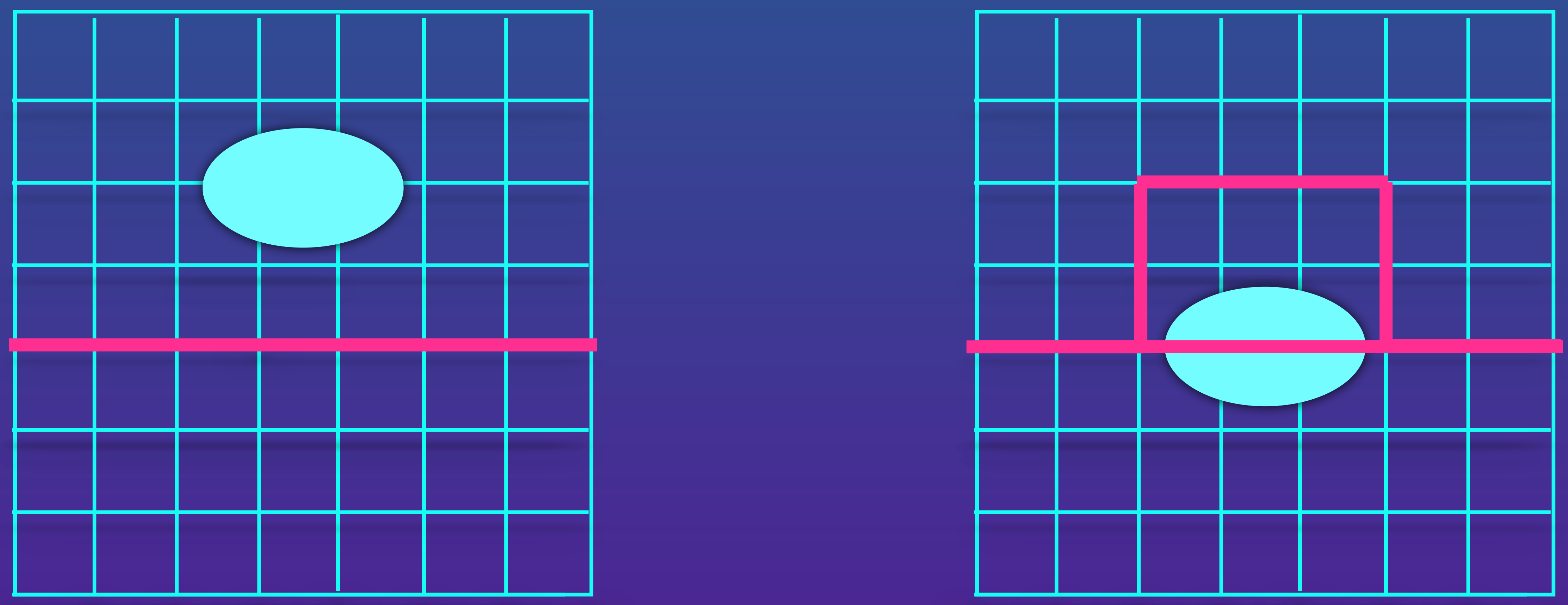
We also achieved goal no. 2

There is a finite Gap

$$H = -\sum_s A_s - \sum_p B_p$$

Can we distinguish the ground states by local observation?

$$Z_1 O = O Z_1$$



Can we distinguish the ground states by local observation?

$$\langle \phi | O | \phi \rangle = \langle \phi ' | O | \phi ' \rangle$$

$$\langle \phi ' | O | \phi ' \rangle = \langle \phi | Z_1 O Z_1 | \phi \rangle = \langle \phi | Z_1 Z_1 O | \phi \rangle = \langle \phi | O | \phi \rangle$$

So we have also achieved goal 3:

The states are not distinguishable locally.

What about goal 4?

What happens if I perturb the Hamiltonian?

$$H \to H + \sum_i O_i$$

$$\Delta E_\alpha = \left\langle \psi_\alpha \middle| \sum_i O_i \middle| \psi_\alpha \right\rangle$$

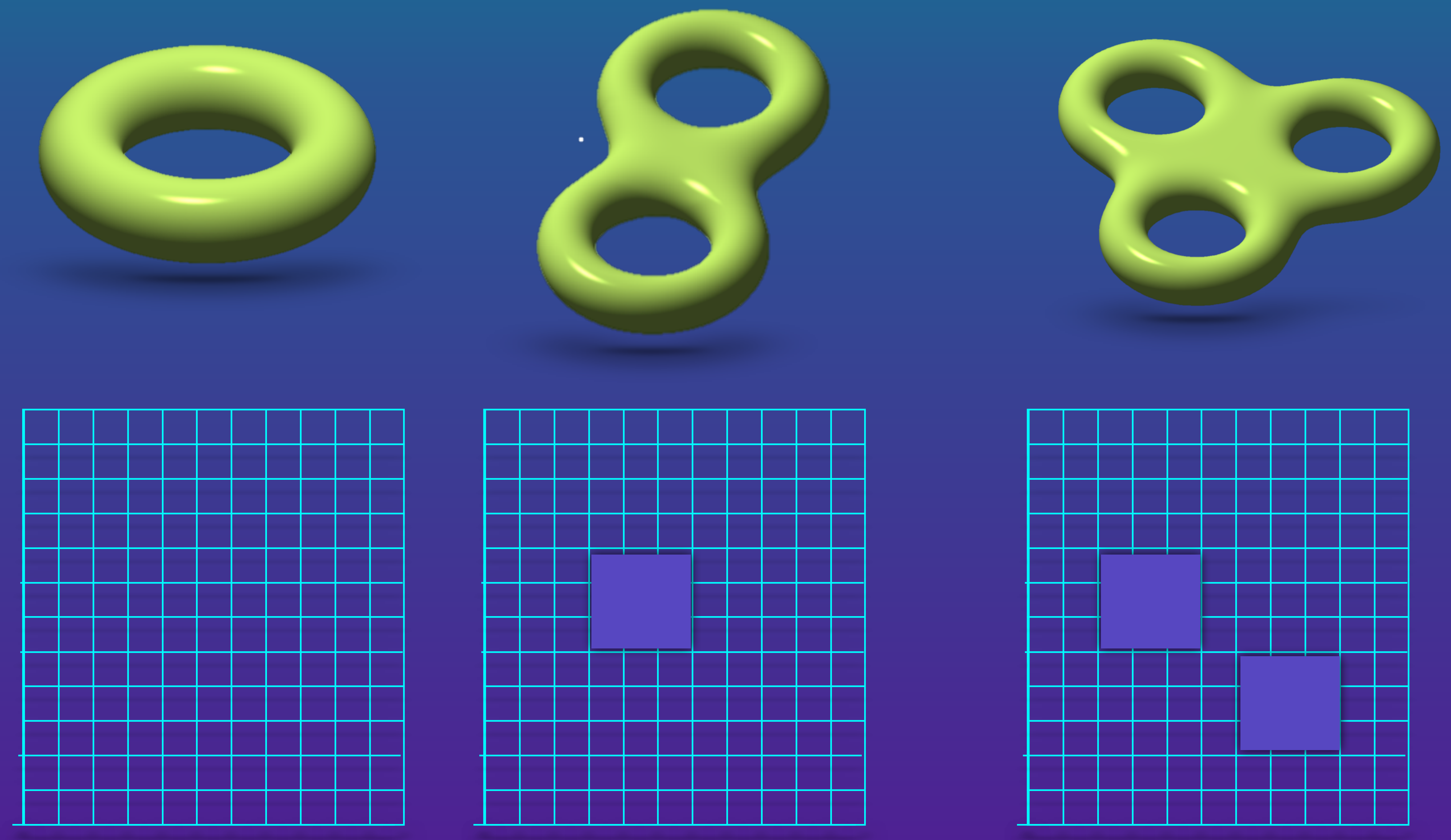
$$\Delta E_\alpha = \Delta E_\beta$$

So we have also achieved goal 4:

The states are robust under perturbations.

We have two topological qubits.

More Qubits



What we have left out:

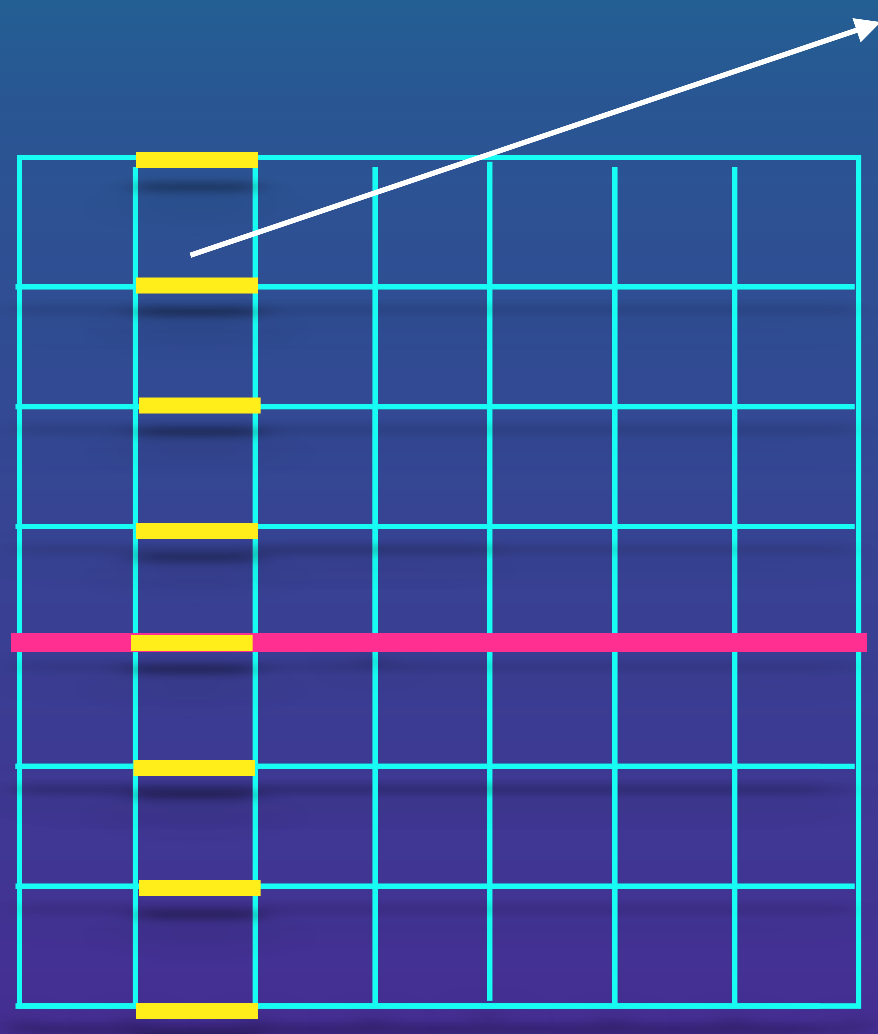
Anyonic Excitations

Moving anyons and performing gates

Non-Abelian Anyons and Universal quantum computation

Thank you for your attention

String operators which distinguish the states!

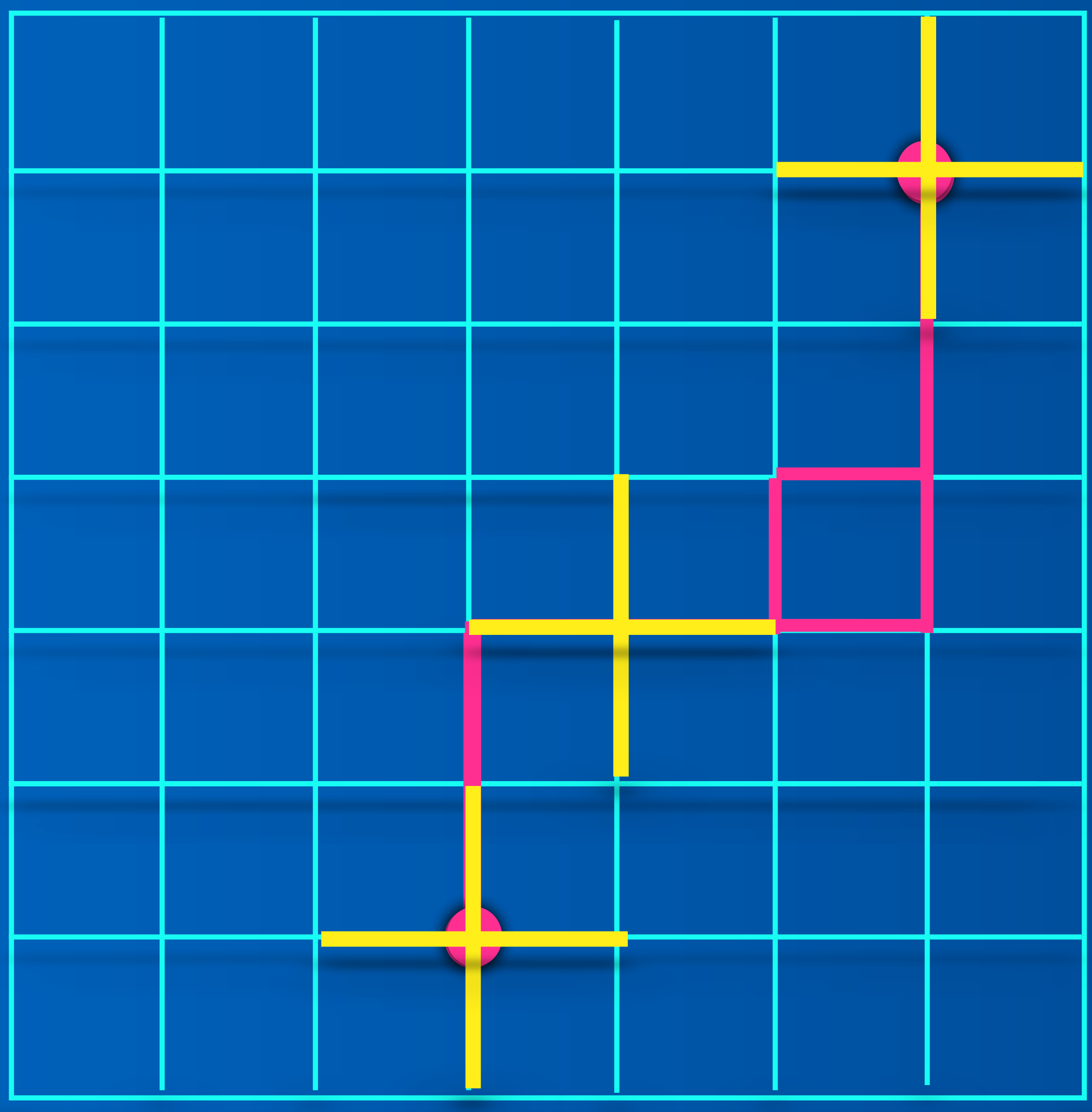


$$X_1 = \prod_{i=1}^{N} x_i$$

$$X_1 Z_1 = -Z_1 X_1$$

$$[X_1, H] = 0$$

Excited States: 1- Electric Anyons.



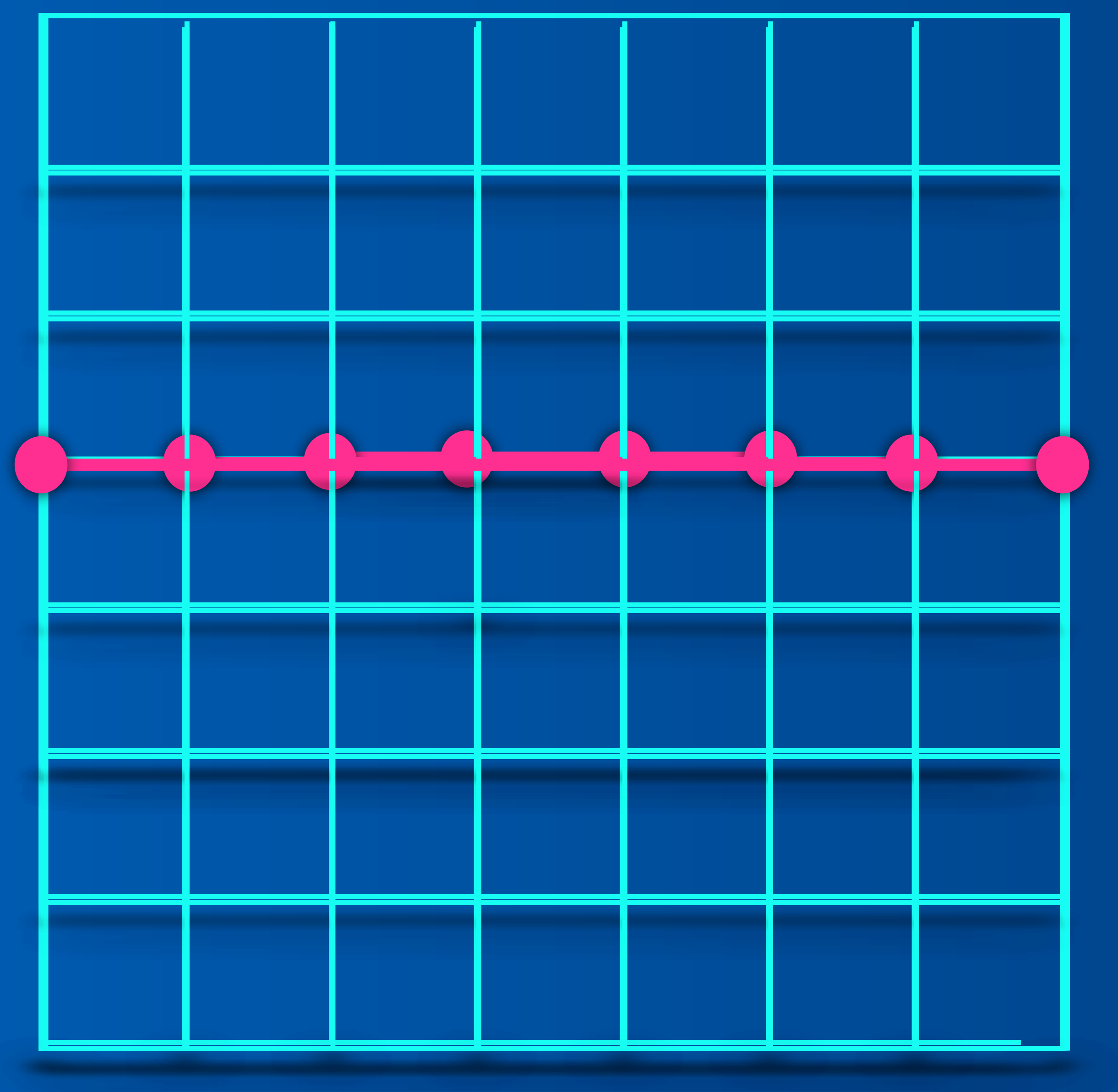
Each Anyon has an energy of

2 units.

Anyons are created in pairs.

The energy of the pair doesn't depend on the path connecting them.

Another interpretation of String Operators

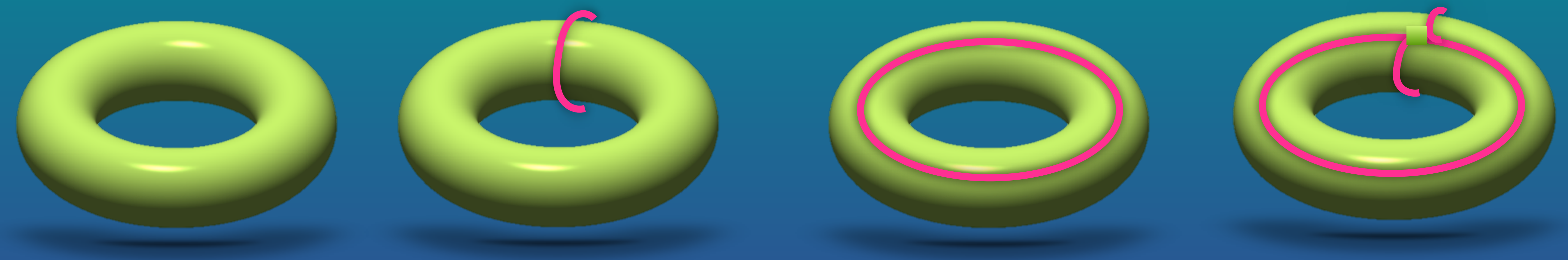


So by creating two electric Anyons,

Moving them across the Torus,

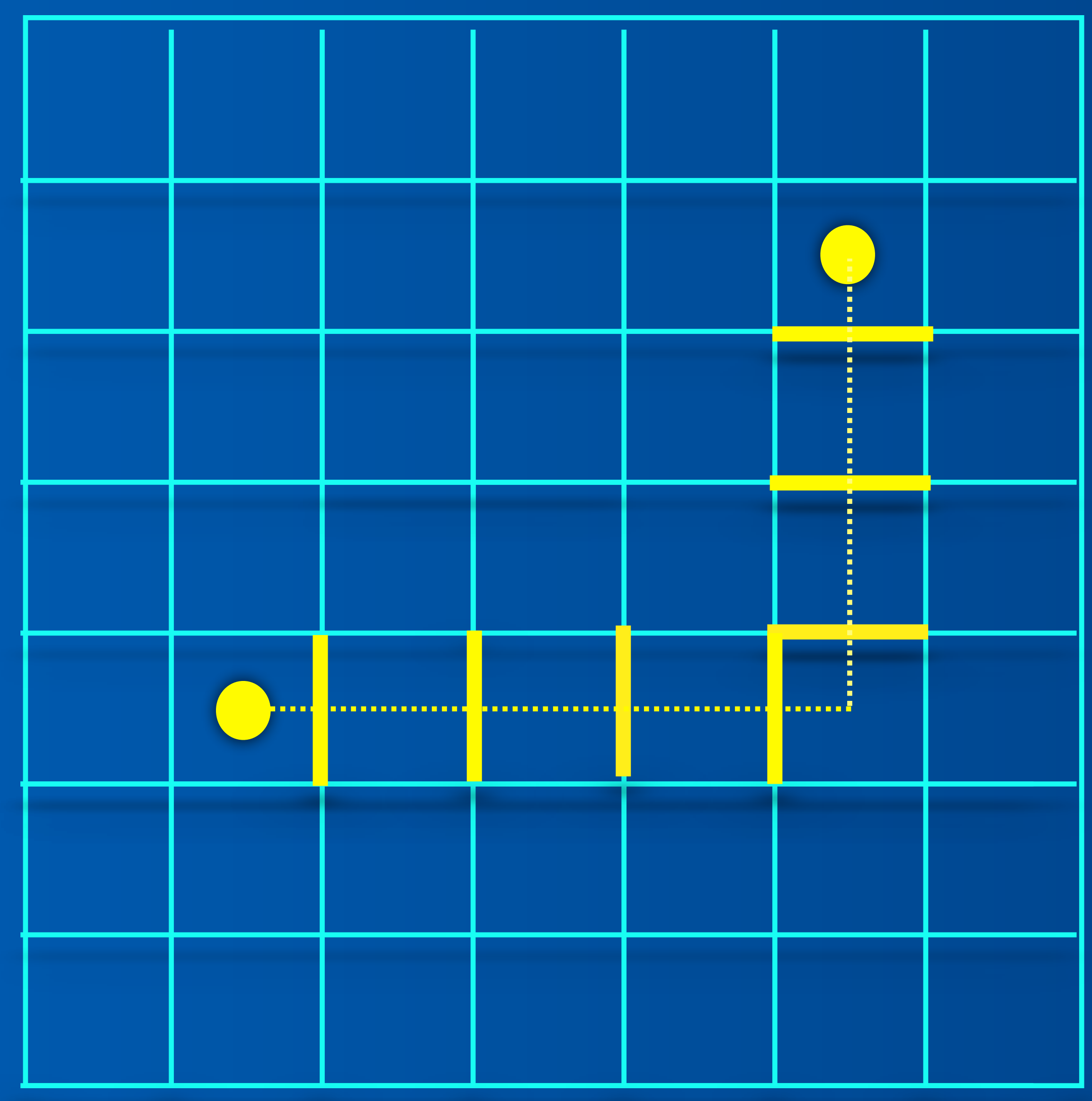
And annihilating them in the end,

We can implement a X gate on either of the qubits.

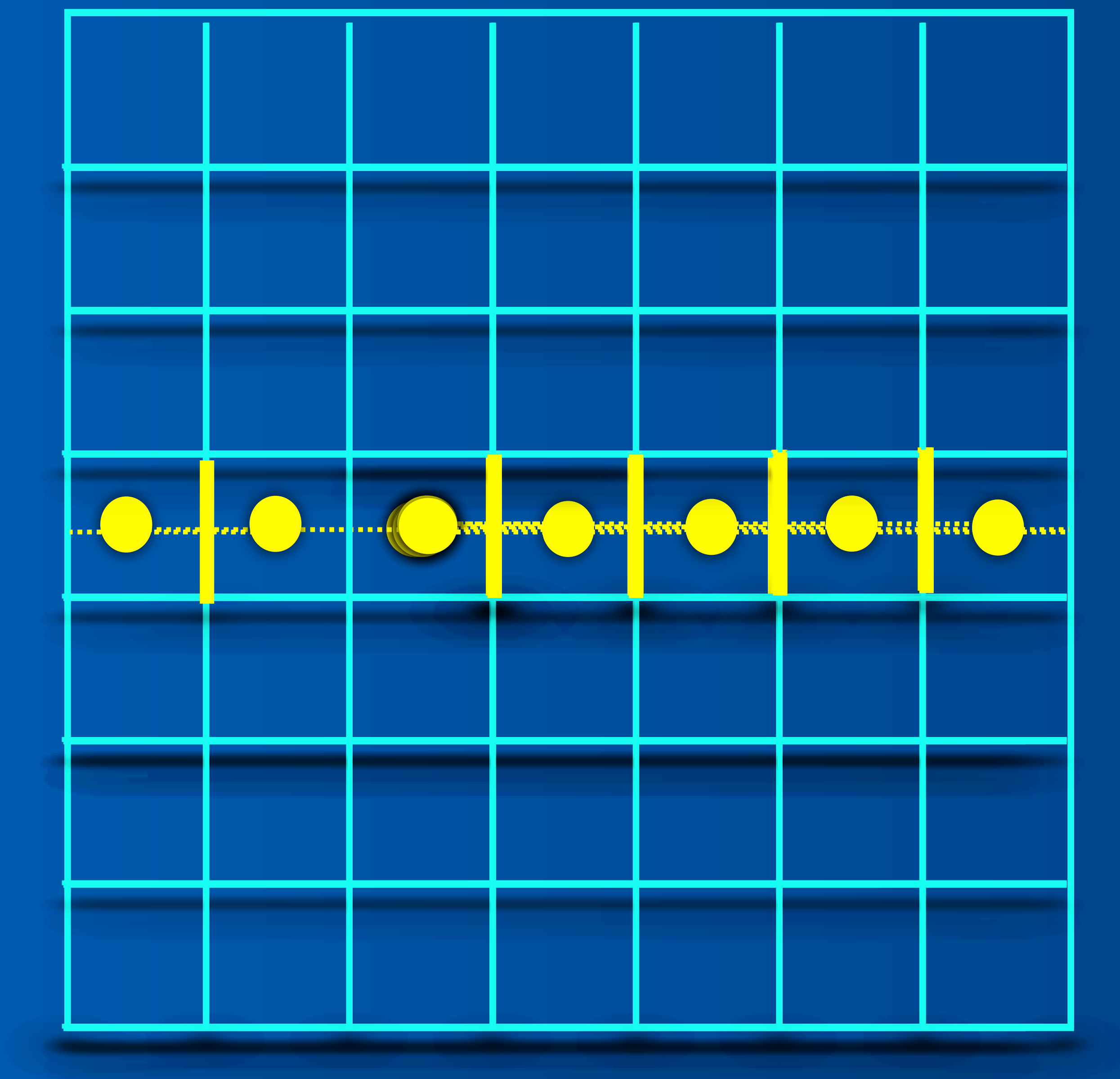


	$\lvert\psi_{00}\rangle$	$\lvert\psi_{10}\rangle = Z_1\lvert\psi_{00}\rangle$	$\lvert\psi_{01}\rangle = Z_2\lvert\psi_{00}\rangle$	$\lvert\psi_{11}\rangle = Z_2 Z_1\lvert\psi_{00}\rangle$
X_1	1	-1	1	-1
X_2	1	1	-1	-1

Excited States: Magnetic Anyons



Another interoperation of string operators



So by creating two Magnetic Anyons,

Moving them across the Torus,

And annihilating them in the end,

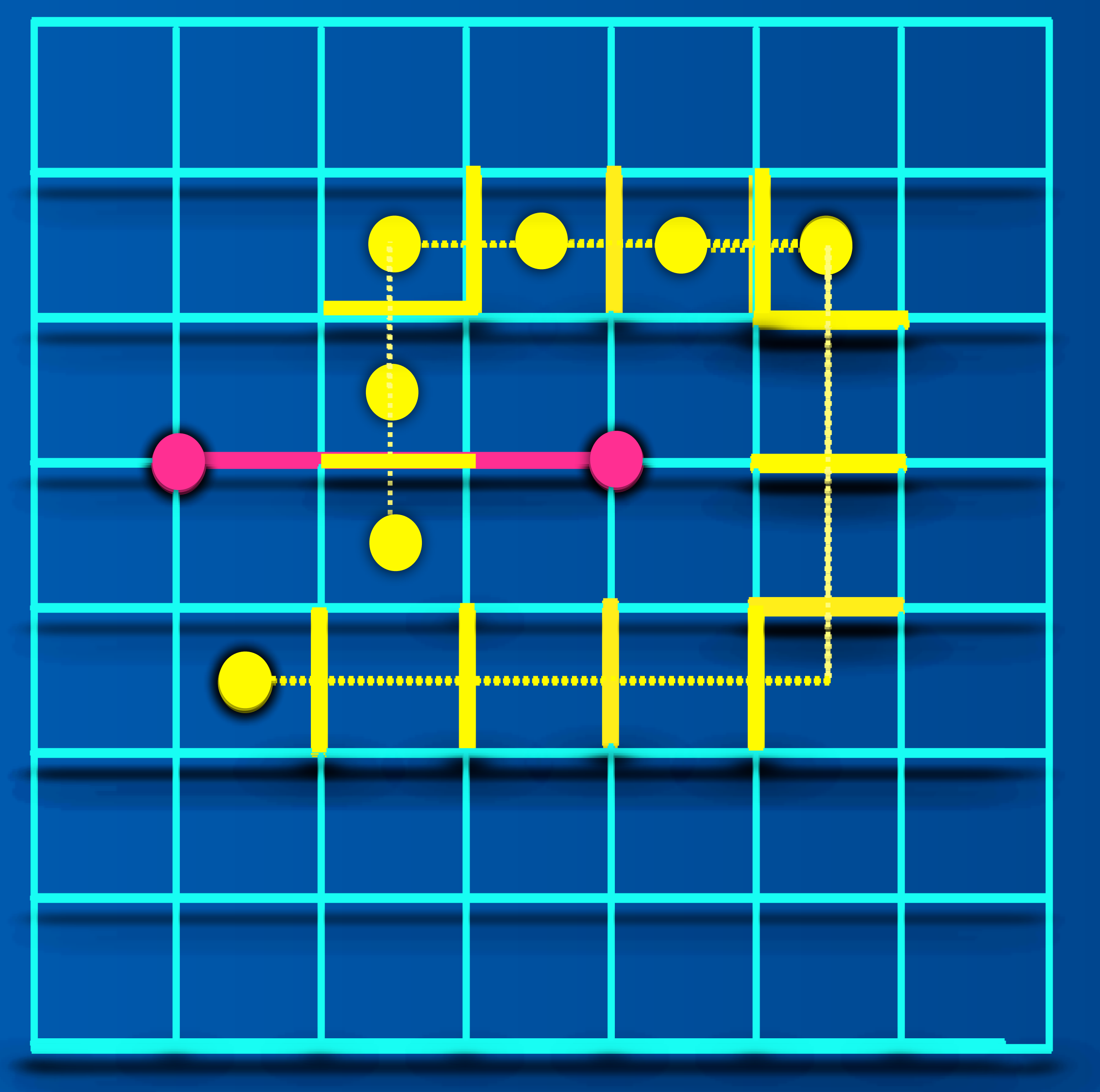
We can implement a Z gate on either of the qubits.

Electric excitations behave as Bosons with respect to each other.

Magnetic excitations behave as Bosons with respect to each other.

But Electric and Magnetic excitations behave as Fermions
with respect to each other.

Why These are Anyons?



So we can do simple,

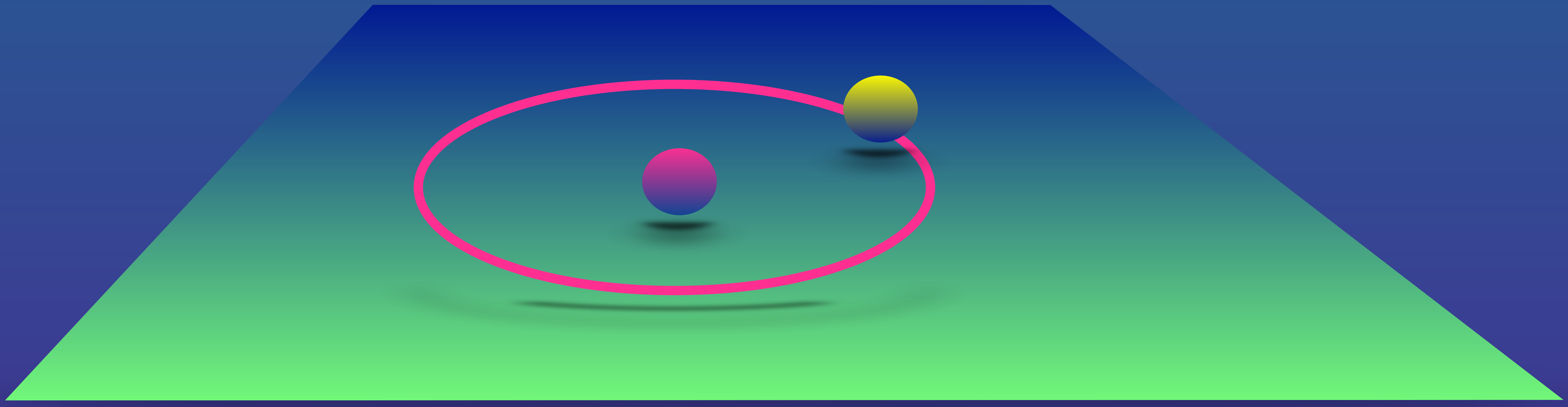
X, Z and Y gates

in a fault-tolerant way.

Unfortunately
the Abelian Models
are not Universal.

We have to consider
Non-Abelian Models.

Non-Abelian Anyons



$$|\psi_i\rangle \rightarrow U_{ij}|\psi_j\rangle$$

Thank you for your attention